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Point Foundation. Lower Bound Solution for Distributed Load

Lars German Hagsten[1]

Introduction

In [1], an upper bound solution for a point foundation subjected to a centrally located load, acting over a finite area with diameter *D*, is derived. In this article, a lower bound solution for the same problem is derived. It is shown that the capacity of this lower bound solution is identical to the capacity of the upper bound solution, and thus an exact solution.

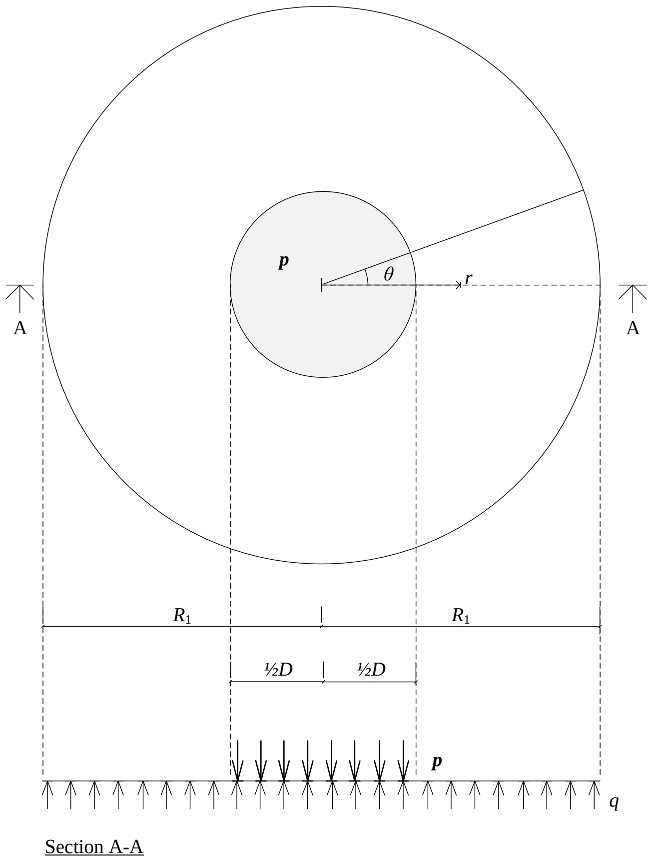

Figure 1

[1] Aarhus University, Value Engineering ApS

For later comparison, the principles and results for an upper and a lower bound solutions for a circular point foundation subjected to a point load, P, are revisited. For further detail see [1].

Figure 2.a shows the mechanism that forms the basis for the upper value solution. The optimal solution is found for $R \rightarrow 0$, The solution is given by:

$$m_\theta = \frac{P}{2\pi} \tag{1}$$

With this solution, only the moment in the yield lines is known, and since the optimum is found for $R \rightarrow 0$, only the moment at the point of the centre is known.

The lower bound value solution is obtained by assuming $m_r = 0$ at every point throughout the foundation. A consequence of this choice is that m_θ can be expressed by:

$$m_\theta = \frac{P}{2\pi}\left(1 - \left(\frac{r}{R_1}\right)^2\right) \tag{2}$$

The maximum value is found at the centre (for $r = 0$), and is given by $m_\theta = \frac{P}{2\pi}$, which is identical to the solution determined by the upper bound solution. As can be seen from the expression, the moment m_θ decreases parabolically towards the edge. The distribution of sectional forces is thus known throughout the foundation. This knowledge of the distribution of the sectional forces can be used, among other things, to investigate whether bent-up reinforcement should be used, and to assess whether the shear capacity is sufficient. Note that the maximum utilization is at the same point and with the same extent for the upper and lower bound solution (one point for foundation subjected to a point load).

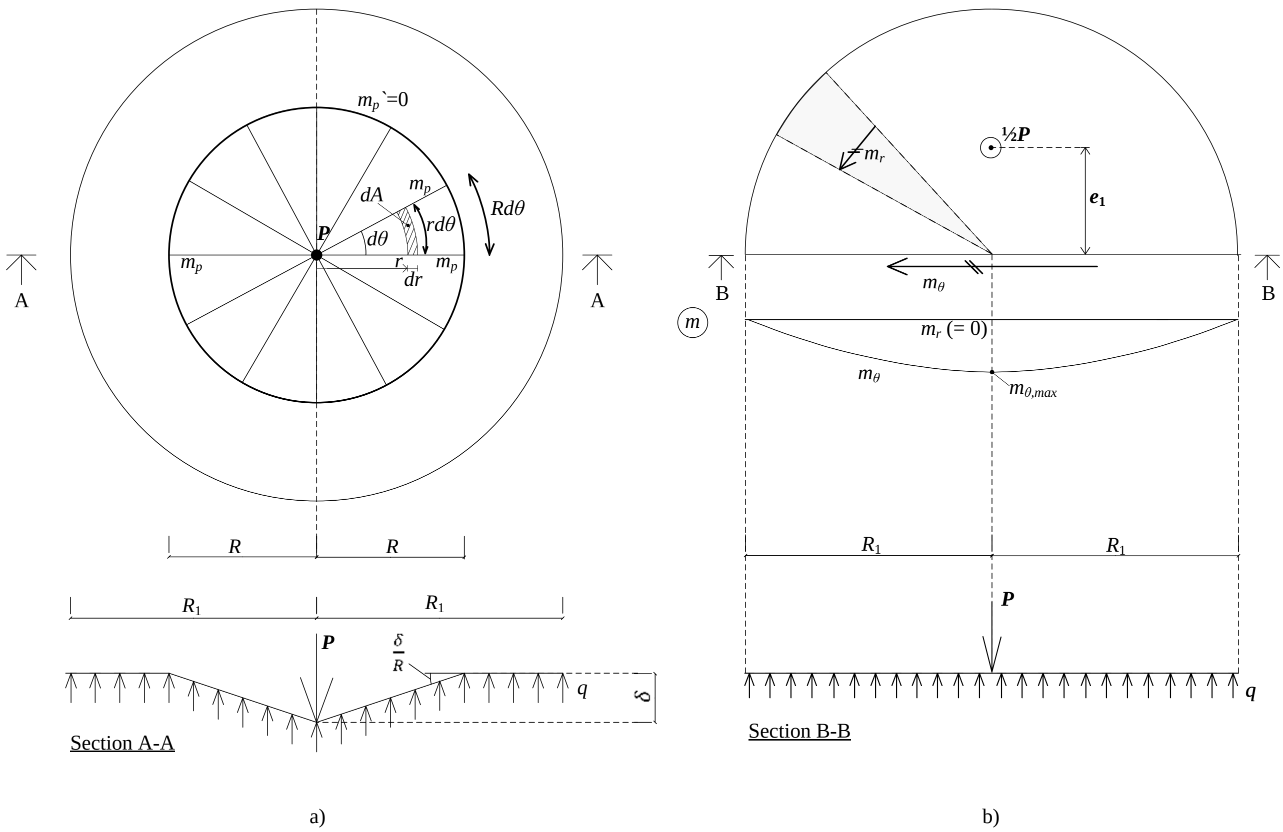

Figure 2. Circular foundation subjected to a point load actin on a finite area.
a) Mechanism (upper bound solution).
b) Moment distribution (lover bound solution)

In [1], an upper bound solution is also derived for a circular point foundation affected by a load acting over a finite area. This area has the diameter D.

Figure 3 shows the critical mechanism. The dimensional moment and thus the required moment capacity is given by:

$$m_p = \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{3}$$

The optimal solution has been found for:

$$R = \sqrt[3]{½D{R_1}^2} \tag{4}$$

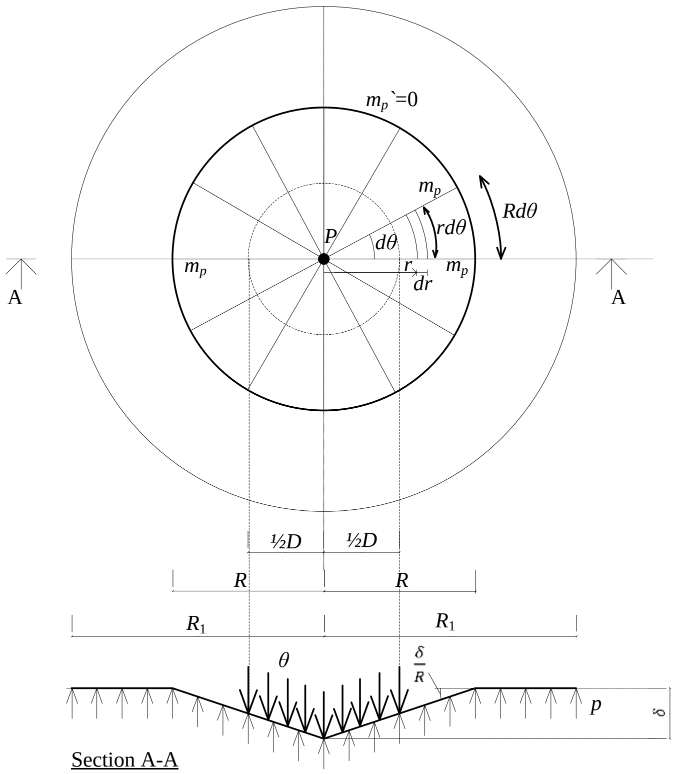

Figure 3. Foundation subjected to a load acting on a finite area. Critical mechanism.

Within the region $r < R$, there dimensional moment is constant. In accordance with the assumed failure mechanism, the moment acts on radial sections (corresponding to m_θ in the lower bound solution). The upper bound solution does not provide information about the moment outside this radius, just as it does not provide information about the magnitude of the moment perpendicular to this moment. In order to obtain information about this, a lower value solution must be sought.

Lower Bound Solution

Figure 4 shows a circular foundation, with radius R_1, subjected to a centrally placed circular surface load over an area with diameter D. The load is called q and the evenly distributed reaction is called p.

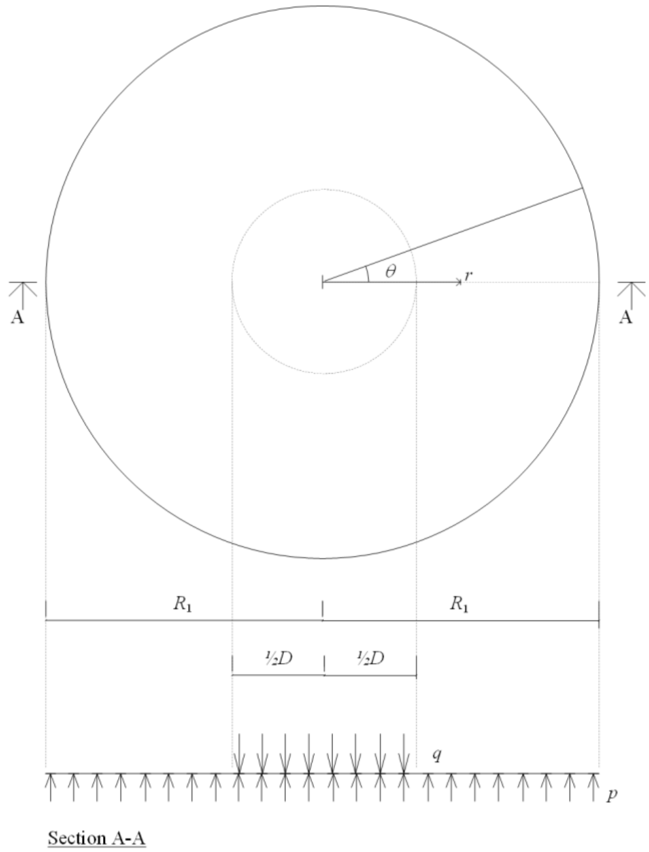

Figure 4 Circular foundation affected by a centrally placed distributed load

Figure 5 shows half of the same foundation. Along the free-cut side through the center is shown the resulting moment, M_θ. The resultant of the reaction is located at the center of gravity of the semicircle located at the distance e_1 from the free-cut side. The resultant of the impact is located at the center of gravity of the semicircle.

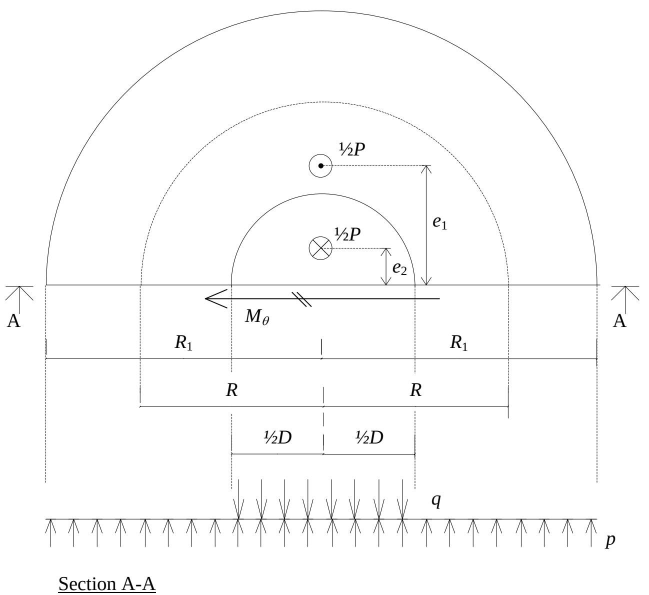

Figure 5 Section through the center of the circular foundations

The resultant of the impact is located at the center of gravity of the semicircle at the distance e_2 from the free-cut side. The magnitude of M_θ is given by:

$$M_\theta = (e_1 - e_2) \cdot \frac{1}{2} P \tag{5}$$

$$M_\theta = \left(\frac{4R_1}{3\pi} - \frac{4(½D)}{3\pi}\right) \cdot \frac{1}{2} P \tag{6}$$

$$M_\theta = \frac{2(R_1 - ½D)}{3\pi} P \tag{7}$$

On the outermost part, at the free edge, the distributed moment, m_θ, is assumed to follow the parabolic curve found during the treatment of the circular foundation affected by point load. The parabolic curve is valid in the region $r \in [R; R_1]$. Within this range, the distributed moment, m_θ, is assumed to be constant, $m_{\theta,1}$. The magnitude of the constant moment is

$$m_\theta(r = R) = \frac{P}{2\pi}\left(1 - \left(\frac{R}{R_1}\right)^2\right) \tag{8}$$

The value of R is found by equilibrium:

$$M_\theta = m_\theta(r = R) \cdot 2R + 2\int_R^{R_1} \frac{P}{2\pi}\left(1 - \left(\frac{r}{R_1}\right)^2\right) dr \tag{9}$$

Of this equation is found:

$$R = \sqrt[3]{½D \cdot {R_1}^2} \tag{10}$$

Inserted into the expression for $m_\theta(r = R)$ the constant distributed moment applicable to is obtained for $r \in [0; R]$:

$$m_{\theta,1} = \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{11}$$

The equilibrium equations [2] are used to check the equilibrium solution, as well as to determine m_r.

1. $v_r \cdot r = \frac{\partial(m_r \cdot r)}{\partial r} - \frac{\partial m_{r\theta}}{\partial \theta} - m_\theta$ (12)
2. $v_\theta = \frac{1}{r}\frac{\partial m_\theta}{\partial \theta} - \frac{\partial m_{r\theta}}{\partial r} - 2\frac{m_{r\theta}}{r}$ (13)
3. $\frac{\partial(v_r \cdot r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta} = -p \cdot r$ (14)
4. $\frac{1}{r}\frac{\partial^2(m_r \cdot r)}{\partial r^2} - \frac{2}{r^2}\frac{\partial^2(m_{r\theta} \cdot r)}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^2 m_\theta}{\partial \theta^2} - \frac{1}{r}\frac{\partial m_\theta}{\partial r} = -p$ (15)

Due to rotational symmetry, $v_\theta = m_{r\theta} = 0$.

As a boundary condition we have that $v_r = m_r = 0$ at the edge.

For $r \in [R; R_1]$ is the solution the same as for the plate with the centrally placed point load and the equilibrium is checked above. The solution is repeated:

$$m_\theta(r) = \frac{P}{2\pi}\left(1 - \left(\frac{r}{R_1}\right)^2\right) \tag{16}$$

$$m_r(r) = 0 \tag{17}$$

$$v_r = -\frac{P}{2\pi r}\left(1 - \frac{r^2}{{R_1}^2}\right) \tag{18}$$

$r \in [½D; R]$

The moment m_θ is constant and determined above:

$$m_\theta = m_{\theta,1} = \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{19}$$

The variation of the shear force is identical to the variation of the shear force for the outer region ($r \in [R_1; R]$):

$$v_r = -\frac{P}{2\pi r}\left(1 - \frac{r^2}{{R_1}^2}\right) \tag{20}$$

The boundary condition for m_r is:

$$m_r(r = R) = 0 \tag{21}$$

m_r is determined by the first equilibrium equation:

$$v_r \cdot r = \frac{\partial(m_r \cdot r)}{\partial r} - \frac{\partial m_{r\theta}}{\partial \theta} - m_\theta \tag{22}$$

Inserted:

$$-\frac{P}{2\pi}\left(1 - \left(\frac{r}{R_1}\right)^2\right) = \frac{\partial(m_r \cdot r)}{\partial r} - \frac{\partial m_{r\theta}}{\partial \theta} - \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{23}$$

$$m_r = \frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{r}{R_1}\right)^2 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) + \frac{c_1}{r} \tag{24}$$

The magnitude of c_1 is determined by the boundary condition for $m_r(r = R_1)$.

$$0 = \frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{R}{R_1}\right)^2 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) + \frac{c_1}{R} \tag{25}$$

$$c_1 = -\frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{R}{R_1}\right)^2 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) R \tag{26}$$

Inserted in the expression for m_r:

$$m_r = \frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{r}{R_1}\right)^2 - \sqrt[3]{\frac{½D}{R_1}}^2 - \left(\frac{1}{3}\left(\frac{R}{R_1}\right)^2 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right)\frac{R}{r}\right) \tag{27}$$

Summarized for $r \in [½D; R]$:

$$m_\theta = \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{28}$$

$$m_r = \frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{r}{R_1}\right)^2 - \sqrt[3]{\frac{½D}{R_1}}^2 - \left(\frac{1}{3}\left(\frac{R}{R_1}\right)^2 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right)\frac{R}{r}\right) \tag{29}$$

$$v_r = -\frac{P}{2\pi r}\left(1 - \left(\frac{r}{R_1}\right)^2\right) \tag{30}$$

$r \in [0; ½D]$

The moment m_θ is constant has the same value as determined above:

$$m_\theta = m_{\theta,1} = \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{31}$$

The shear force is given by:

$$v_r = -\frac{P}{2\pi r}\left(1 - \left(\frac{r}{R_1}\right)^2\right) + \frac{P}{2\pi r}\left(1 - \left(\frac{r}{½D}\right)^2\right) \tag{32}$$

$$v_r = \frac{P}{2\pi r}\left(\frac{1}{{R_1}^2} - \frac{1}{(½D)^2}\right) r^2 \tag{33}$$

The boundary condition for m_r is:

$$m_r(r = R_0) = \frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{½D}{R_1}\right)^2 - \sqrt[3]{\frac{D^2}{4{R_1}^2}} - \left(\frac{1}{3}\left(\frac{R}{R_1}\right)^2 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right)\frac{R}{½D}\right) \tag{34}$$

m_r is again determined by the first equilibrium equation:

$$v_r \cdot r = \frac{\partial(m_r \cdot r)}{\partial r} - \frac{\partial m_{r\theta}}{\partial \theta} - m_\theta \tag{35}$$

$$\frac{P}{2\pi}\left(\frac{1}{{R_1}^2} - \frac{1}{(½D)^2}\right)r^2 = \frac{\partial(m_r \cdot r)}{\partial r} - \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{36}$$

$$m_r = \frac{P}{2\pi}\left(\left(\frac{1}{3}\frac{1}{{R_1}^2} - \frac{1}{3}\frac{1}{(½D)^2}\right)r^2 + 1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) + \frac{c_2}{r} \tag{37}$$

c_2 is determined by the boundary condition for m_r:

$$\frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{½D}{R_1}\right)^2 - \sqrt[3]{\frac{½D}{R_1}}^2 - \left(\frac{1}{3}\left(\frac{R}{R_1}\right)^2 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right)\frac{R}{½D}\right) = \frac{P}{2\pi}\left(\left(\frac{1}{3}\frac{1}{{R_1}^2} - \frac{1}{3}\frac{1}{(½D)^2}\right)r^2 + 1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) + \frac{c_2}{r} \tag{38}$$

$$c_2 = \frac{P}{2\pi}\left(\sqrt[3]{\frac{½D}{R_1}}^2 \frac{R}{½D} - \frac{1}{3}\frac{R^3}{{R_1}^2(½D)} - \frac{2}{3}\right)½D \tag{39}$$

With the expression for R_1 indserted:

$$c_2 = 0 \tag{40}$$

Inserted:

$$m_r = \frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{1}{{R_1}^2} - \frac{1}{(½D)^2}\right)r^2 + 1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{41}$$

Summarized for $r \in [0; ½D]$:

$$m_\theta = \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{42}$$

$$m_r = \frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{1}{{R_1}^2} - \frac{1}{(½D)^2}\right)r^2 + 1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{43}$$

$$v_r = \frac{P}{2\pi r}\left(\frac{1}{{R_1}^2} - \frac{1}{(½D)^2}\right)r^2 \tag{44}$$

2. Summary of lower bound solution

Boundaries/crossings:
D is the diameter of the column
R_1 is the radius of the foundation
R represents the distance at which m_θ changes from constant to decreasing moment, and is given by: $R = \sqrt[3]{½D \cdot {R_1}^2}$

$r \in [0; ½D]$:

$$m_\theta = \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{½D}{R_1}}^2\right) \tag{45}$$

$$m_r = \frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{1}{{R_1}^2} - \frac{1}{(½D)^2}\right)r^2 + 1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{46}$$

$$v_r = \frac{P}{2\pi r}\left(\frac{1}{{R_1}^2} - \frac{1}{(½D)^2}\right)r^2 \tag{47}$$

$r \in [½D; R]$:

$$m_\theta = \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{48}$$

$$m_r = \frac{P}{2\pi}\left(\frac{1}{3}\left(\frac{r}{R_1}\right)^2 - \sqrt[3]{\frac{D^2}{4{R_1}^2}} - \left(\frac{1}{3}\left(\frac{R}{R_1}\right)^2 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right)\frac{R}{r}\right) \tag{49}$$

$$v_r = -\frac{P}{2\pi r}\left(1 - \left(\frac{r}{R_1}\right)^2\right) \tag{50}$$

$r \in [R; R_1]$:

$$m_\theta(r) = \frac{P}{2\pi}\left(1 - \left(\frac{r}{R_1}\right)^2\right) \tag{51}$$

$$m_r(r) = 0 \tag{52}$$

$$v_r = -\frac{P}{2\pi r}\left(1 - \frac{r^2}{{R_1}^2}\right) \tag{53}$$

With respect to the variation of m_θ and m_r see Figure 6.

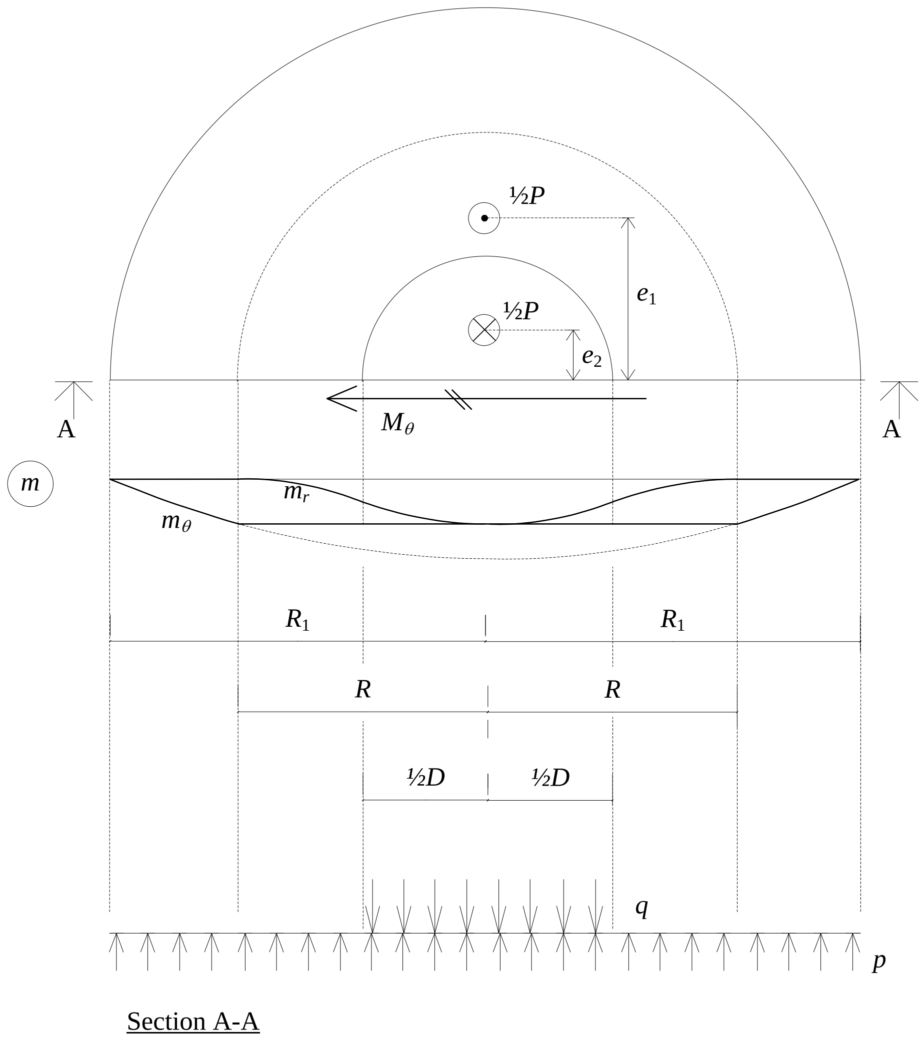

Figure 6 Distribution of m_θ and m_r

It can be seen that the maximum moment is given by:

$$m_\theta = m_r = \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4{R_1}^2}}\right) \tag{54}$$

By comparing with the upper bound solution from [1] for the same problem, it can be seen that it is an exact solution.

The distance at which m_θ changes from constant to the parabolic is seen to be identical to the bounding of the mechanism corresponding to the upper bound solution, see formula (4) and formula (10). For m_θ shown in

Figure 6 the dotted thin line shows the moment in the case of the load modelled as appoint load. The difference between this dotted line and the fully marked, line, showing a constant value within the area limited by R, illustrates the reduction of m_θ by taking the extend of the load (finite area with diameter D) acting on the foundation into account.

The advantage of the lower bound solution is also that the sectional forces are known throughout the foundation. This knowledge can be used to assess whether there is a need to anchor the reinforcement at the edge, which is typically done by bending up the reinforcement at the edge.

Conclusion

The article presents a lower bound solution for a circular foundation subjected to a centrally located circular surface load. The solution is an exact solution as it gives a capacity identical to the upper bound solution found in [1].

The expressions for the moments and shear forces vary in different areas of the foundation.

The lower bound solution provides information on the variation of the sectional forces throughout the foundation, including the variation towards the edge. This information can be used to assess the need to anchor the reinforcement at the edge of the foundation.

Reference

[1] Lars German Hagsten, Merle Rianne van Logtestijn, Henning Højgaard Laustsen: ”*Point foundation. Upper and lower bound solution for point load and upper bound solution for distributed load*”. Proceedings of the Danish Society for Structural Science and Engineering. No. 2, Nov 2022, page 1-14.

[2] M.P. Nielsen & L. C. Hoang: 'Limit Analysis and Concrete Plasticity'. Third edition. 2011. CRC Press

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