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Strain Capacity of Reinforced Concrete Disks

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Abstract

The yield condition for disks is a very rational method for designing concrete disks and for determining the strength of disks for arbitrary load combinations. When determining the yield condition for disks, use is made of an ideal plastic material modelling.

In makes sure that the corresponding strain capacity is sufficient, the ideal plastic material modelling is replaced with a bilinear stress strain curve for the reinforcement, whereby a clear connection between strain and tension is established.

Due to the interaction between concrete and reinforcement, the strain capacity of the embedded reinforcement is smaller than the strain capacity of the bar reinforcement.

General limits for the choice of anisotropy ratio of the reinforcement so that sufficient strain capacity in the reinforcement is present by any combinations of normal and shear stresses is presented. Limits are set for the choice of maximum anisotropy ratio for three different reinforcement types.

By adding limits on the maximum allowable degree of anisotropy as function of the lowest degree of reinforcements to the application of the yield condition, it is ensured that the strain capacity of the reinforcing is sufficient for any combination of the stresses obeying the yield condition.

Introduction

For disks subjected by shear, τ, the yield condition is given by:

$$|\tau_{xy}| \leq \sqrt{(f_{tx} - \sigma_x)(f_{ty} - \sigma_y)} \qquad (1)$$

[1] Aarhus University

Any combination of σ_x, σ_y and τ_{xy}, that satisfies Equation 1 can be carried by the disk. f_{tx} is the tensile capacity in the direction of the *x*-axis just as f_{ty} is the tensile capacity in the direction of the *y*-axis. The yield condition was derived by MPN [] and is incorporated in EC2 [].

The yield condition has the origin in the theory of plasticity and is based on a rigid plastic material modelling. The modelling has shown being strong in predicting the capacity. In Figure 1 is seen a comparison between tests and modelled capacity.

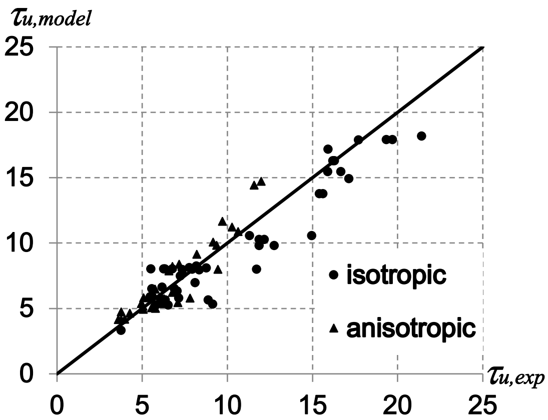

Figure 1 Comparison between model and shear tests

As the yield condition is based on a rigid plastic material modelling, no information is available on the strains in neither concrete or reinforcement whereby no information is available on whether the strain capacity of the materials is sufficient.

In order to achieve such information on the strain state a bilinear material modelling of the reinforcement is adopted and replacing the rigid plastic modelling of the reinforcement.

Kauffmann [] has made a corresponding investigation on the strain capacity of the concrete. Focus in this paper is on the strain capacity of the reinforcement.

The yield condition is developed under the assumptions of (i) no tensile capacity of the concrete (ii) orthogonal reinforcement (iii) shear is carried by a combination of diagonal compression in concrete being in equilibrium with tension in the reinforcement and/or applied normal stresses (iv) a rigid plastic material modelling.

Based on the first three assumptions and with no matter of the material modelling the stress state in the material is given by:

$$\sigma_c = |\tau_{xy}|\left(\tan\theta + \frac{1}{\tan\theta}\right) \quad (2)$$

$$\sigma_{sx} = \left(\sigma_x + \frac{|\tau_{xy}|}{\tan\theta}\right)\frac{1}{\rho_x} \quad (3)$$

$$\sigma_{sy} = \left(\sigma_y + |\tau_{xy}|\tan\theta\right)\frac{1}{\rho_y} \quad (4)$$

Where σ_c is the compressive stress in the concrete, σ_{sx} and σ_{sy} are the stresses in the reinforcement respectively in the *x*- and *y*-direction. ρ_x and ρ_y are the geometrical degree of reinforcement in the *x*- and *y*-direction. θ is the orientation of the diagonal compression stress in concrete, see Figure 2. For known geometry, degrees of reinforcement and applied loads (σ_x, σ_y and τ_{xy}), the only unknown parameter is the orientation of the diagonal compression stress, θ.

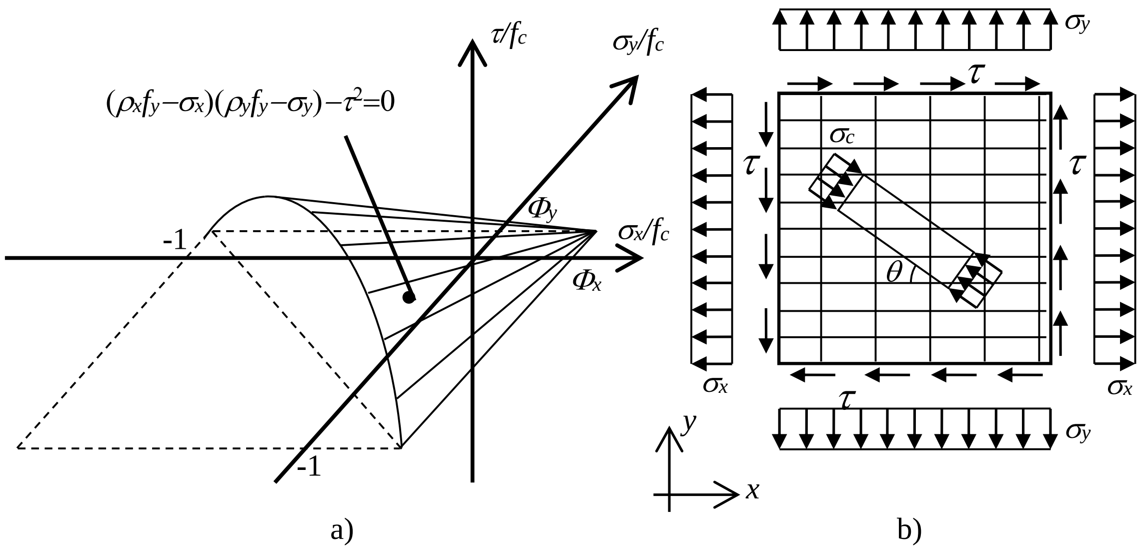

Figure 2 a) Illustration of yield condition. b) Geometry, layout of reinforcement and stresses at the boundaries

The magnitude of the inclination is a function of the ratio between the applied stresses and on the ratio between the degree of reinforcement in the two directions. For pure shear the compression in the concrete orient towards the largest degree of reinforcement and due to equilibrium leading to the largest forces in this direction. But at stress and strain level, the largest stresses and strains will be in the reinforcement in the direction with the smallest degree of reinforcement. This is due to the fact that the ratio between the forces in the two direction caused by the inclination of the compression in the concrete is smaller than the ratio between the degree of reinforcement in the two directions. At the load level at which yielding in the reinforcement in the direction of the smallest degree of reinforcement is initiated, the inclination of the compression in the concrete is further oriented towards the direction with the largest degree of reinforcement. Regardless of that, the magnitude of the strain in the direction of the smallest degree of increases most and thereby becomes crucial with respect to strain capacity. Again this is due to the fact that the ratio between the forces in the two direction is smaller than the ratio between the degree of reinforcement in the two directions.

Under the assumption of a rigid plastic material modeling and yielding in the reinforcement in both directions θ is found by setting $\sigma_{sx} = \sigma_{sy} = f_y$ and solving equation (3) and (4) with respect to θ.

In a pure elastic modeling, θ is found by minimizing the strain energy in the disk.

In the case of a bilinear material modeling the non-conservative contribution from the inelastic behavior must simultaneously by subtracted in the minimization of the potential energy [LGH].

The average strain in an embedded reinforcement bar is affected by tension stiffening. For small degrees of reinforcement the effect of tension stiffening becomes most significant. That means that the strain capacity, seen as the average strain level in the reinforcement when f_u is reached in the reinforcement in the cracks, is smallest for the smallest degrees of reinforcement. This can be realized as smaller degrees of reinforcement leads to larger crack distances.

As both crack distances and bond strength between concrete and reinforcement is a function of concrete strength, the restriction may also include information on the concrete strength.

The restrictions can therefore be expected to be a function of (i) the smallest degree of reinforcement (ii) the degree of anisotropy and (iii) the concrete strength.

The aim of this work are to:

1) - Set up a general expression in order to determine the orientation of the diagonal stress in the concrete and thereby be able to determine the actual stresses/strains in the reinforcement.

2) - For different kind of ductility classes of the reinforcement to determine the maximum allowable degree of anisotropy as function of the lowest degree of reinforcements.

By adding limits on the maximum allowable degree of anisotropy as function of the lowest degree of reinforcements to the application of the yield condition, it is ensured that the strain capacity of the reinforcing is sufficient for any combination of the stresses obeying the yield condition given by formula (1).

The aim of this paper is therefor to examine how the application of the yield condition shall be complemented with restrictions ensuring sufficient strain capacity.

The approach is initially illustrated by two test reported by Vecchio and Collins [1982]. The disks were both affected by pure shear, τ_{xy} ($\sigma_x = \sigma_y = 0$).

The disks had degrees of reinforcement given by respectively ρ_x/ρ_y (= ρ_{max}/ρ_{min}) = 0.0181/0.0072 = 2.5 and ρ_x/ρ_y = 0.0181/0.0032 = 5.7. As an outset both disks were uncracked whereby the diagonal concrete stress, σ_c, is oriented at an angle of 45°. At a certain load level the disks will crack.

For a load larger than the cracking load, and less than the load level at which the reinforcement yields, the orientation of the stress in the concrete will be determined by minimizing the strain energy. Hereby it is found that the orientation of the stress in the concrete will change so the stress is oriented towards the direction with the largest degree of reinforcement. That is, θ is reduced for load levels (here expressed by τ_{xy}) higher than the cracking load. By analyzing the stresses in the reinforcement at this load level, it is seen that the stresses are largest in the direction with the lowest degree of reinforcement.

By further uploading, yielding in the direction with the lowest degree of reinforcement will be reached at a certain load level. From the load level at which the disk crack and up to the load level where yielding occurs, the orientation of the stress in the concrete is constant and consequently the stress in the reinforcement is proportional to the load level. At loading above the load level at which yielding occurs, the reinforcement in the direction of the lowest degree of reinforcement will yield and the inclination of the compression in the concrete will further orientate towards the direction with the largest degree of reinforcement.

The modeled inclination of the principal stress in the concrete and the maximum strain in the reinforcement is sketched in figure 3 as function of the applied load for both disks.

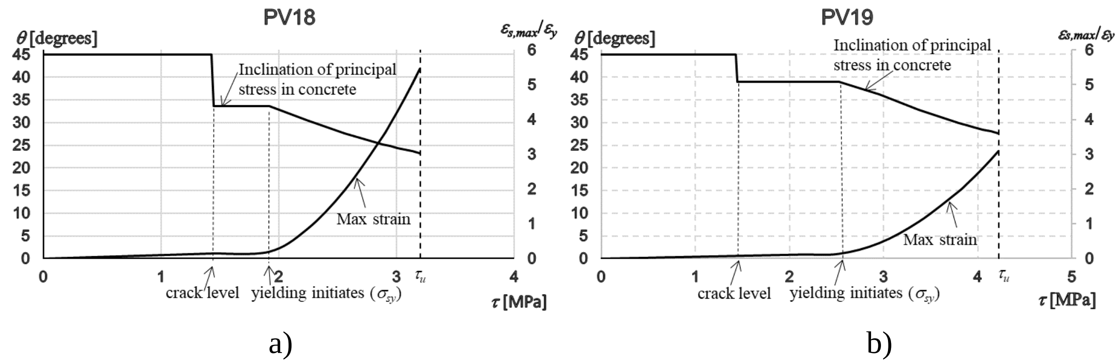

Figure 3 Modeled inclination of the principal stress in the concrete and the maximum strain in the reinforcement for PV 18 ad PV19

In PV19, the ratio between the max. strain in the reinforcement and the yield strain is 3.1. In PV18, the ratio between the max. strain in the reinforcement and the yield strain is 5.5. It is a general observation, that larger ratios of ρ_x/ρ_y leads to larger needed strain capacities of the reinforcement in order to reach the capacity given by (1).

2 Theory

2.1 Equilibrium for inelastic materials

A state of equilibrium can generally be determined by minimizing the potential energy. This ensures that both the static and geometric conditions are fulfilled. For statically indeterminate systems made of linear elastic materials, equilibrium is found by minimizing the elastic energy. For systems with materials exhibiting inelastic behavior, equilibrium can be determined by minimizing the potential energy taking into account energy dissipation [2017].

The potential energy is determined as the sum of the elastic energy stored in the materials, $\Pi_{elastic}$ subtracted the product of the external forces and their displacements, Π_P. Owing to the inelastic material behavior, the system also loses mechanical/potential energy, Δw. This part must subtracted the potential energy, Π.

$$\Pi - \Delta w = \Pi_{elastic} - \Pi_P$$

$$\Pi = \Pi_{elastic} - \Pi_P + \Delta w \qquad (5)$$

Equilibrium is the determined by requiring:

$$\frac{d\Pi}{d\theta} = 0 \qquad (6)$$

In order to be able to express the different contributions to the corrected potential energy, knowledge of constitutive relations as well as the displacements of the boundaries is needed.

2.2 Material modeling

2.2.1 Concrete

The concrete is assumed to have no tensile strength, and in regards to compression, a linear elastic-ideal plastic behavior is assumed. The following relations between stresses and strains are applied:

$$\sigma_c = \begin{cases} \varepsilon_c E_c & \text{for } \varepsilon_c \leq \varepsilon_{co} \\ \nu f_c & \text{for } \varepsilon_c \geq \varepsilon_{co} \end{cases} \tag{7}$$

E_c is taken as the average secant-modulus and ν is the effectiveness factor. Both are determined according to Eurocode 2 [2008]:

$$\nu = 0.6\left(1 - \frac{f_{ck}}{250}\right) \tag{8}$$

$$E_c = 22\left(\frac{f_{cm}}{10}\right)^{0.3}, f_{cm} = f_{ck} + 8 \tag{9}$$

2.2.2 Naked and embedded Reinforcement

The naked reinforcement is modelled using a bilinear stress-strain curve, as shown in Figure 5a.

Due to the bond between concrete and reinforcement, the strain varies along the embedded reinforcement. Peak steel strains are obtained at the position of cracks, and in-between cracks the strains are reduced owing to the restraining action of bond.

Different approaches in regards to modelling of reinforcement bond has been proposed [1998], [2007], [2011]. In the present approach, a slightly corrected version the Tension Chord Model, proposed by Marti *et al.* [1998] is adopted. According to the Tension Chord Model, the crack distance in reinforced concrete bar is given as:

$$s_{rm} = 1.33\frac{\text{ø}}{8}\frac{1-\rho_s}{\rho_s} \tag{10}$$

Figure 4 shows a comparison between measured crack distances and the crack distance determined by formula (10) as a function of the degree of reinforcement.

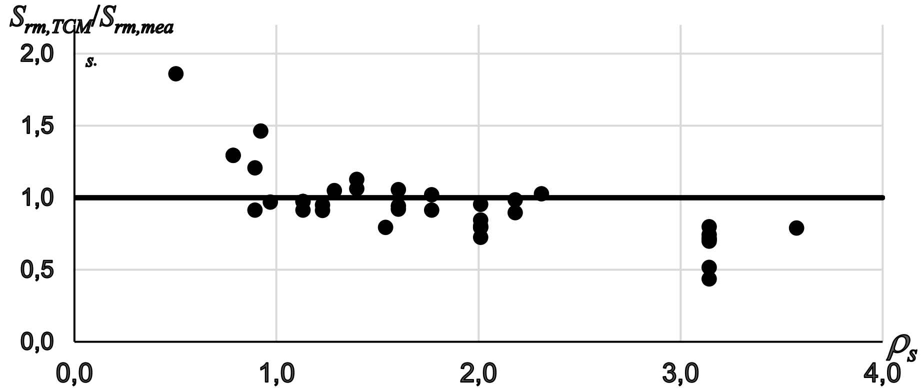

Figure 4 Comparison between measured crack distances and the crack distance determined by the Tension Chord Model as function of ρ_s

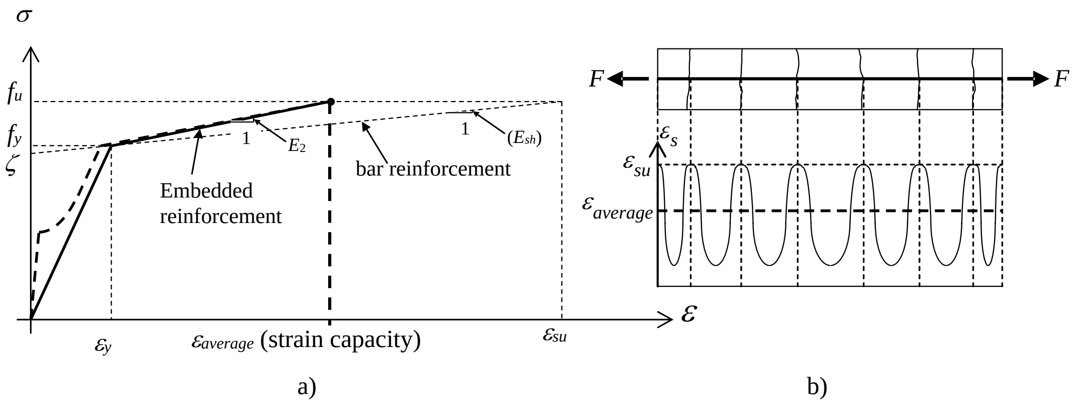

Figure 5 a) Model of embedded and bar reinforcement b) Variation of strains in embedded reinforcement

Locally, in a zone near the cracks visible at the surface, small conical cracks [1971] develops, and, as a consequence, a zone with no bond develops near the cracks. The influence of such local effects has also been recognized by Leoanhardt []. In an effort to take the effect of such local zones with no/limited bond into account Jokela [1986] set up an expression for the length of the debonded zone near the cracks. This is given by:

$$l_{deb} = \frac{1}{2}\left(1 + \frac{\sigma_s}{100}\right) \text{ø}_s \tag{11}$$

The aim of the present investigation is to model the case where the stresses in the cracks has reached f_u. When evaluating l_{deb} the reinforcement stress will be taken as f_u as a reasonable simplification.

Next to the debonded zones, reinforcement and concrete are interacting in accordance to the Tension Chord Model. Accordingly, the determination of bond stresses is done using two different expressions; one for case where the reinforcement behaves elastically, τ_e, and one for case where the reinforcement is yielding, τ_y:

$$\tau(x) = \begin{cases} \tau_e = 0.6 {f_c}^{2/3} \\ \tau_y = 0.3 {f_c}^{2/3} \end{cases} \tag{12}$$

The strain capacity of the embedded reinforcement can be determined as the average strain in the reinforcement in the event that the ultimate strain of the reinforcement is reached in the cracks.

The strain capacity can be determined as the average strain in the reinforcement over a distance eual to the distance between two cracks, s_{rm}:

$$\varepsilon_{s,capacity} = \varepsilon_{s,average} = \frac{1}{s_{rm}} \cdot 2 \int_0^{½s_{rm}} \varepsilon_s(x)dx \tag{13}$$

An expression for the variation of the strain in the reinforcement in the neighborhood of a crack (here $x = 0$ at the crack) will accordingly be divided into three parts:

$$\varepsilon(x) = \begin{cases} \varepsilon_u & \text{for} \quad x \leq l_{deb} \\ \frac{1}{E_2}\left(f_u - \zeta - \frac{4\tau_y}{ø_s}(x - l_{deb})\right) & \text{for } l_{deb} < x \leq l_p \\ \frac{1}{E_s}\left(f_y - \frac{4\tau_e}{ø_s}(x - l_p)\right) & \text{for } l_p < x \end{cases} \tag{14}$$

l_p is the distance from the crack to the location where $\sigma_s = f_y$. This distance is given by:

$$l_p = \frac{(f_u - f_y)ø_s}{4\tau_y} + l_{deb} = \frac{(f_u - f_y)ø_s}{4\tau_y} + \frac{1}{2}\left(1 + \frac{f_y}{100}\right)ø_s \tag{15}$$

Inserting (14) in (13) leads to the following expressions for $\varepsilon_{average}$ depending on the length of the plastic zone.

If $½s_{rm} > l_p$:

$$\varepsilon_{average} = \frac{1}{s_{rm}} 2 \left(\begin{array}{c} \varepsilon_u l_{deb} + \frac{1}{E_2}\left(\left(f_u - \zeta + \frac{4\tau_y}{ø_s} l_{deb}\right)(l_p - l_{deb}) - \frac{2\tau_y}{ø_s}\left({l_p}^2 - {l_{deb}}^2\right)\right) \\ + \frac{1}{E_s}\left(f_y + \frac{4\tau_e}{ø_s} l_p(½s_{rm} - l_p) - \frac{2\tau_e}{ø_s}\left(\frac{1}{4}{s_{rm}}^2 - {l_p}^2\right)\right) \end{array} \right) \tag{16a}$$

If $½s_{rm} < l_p$:

$$\varepsilon_{average} = \frac{1}{s_{rm}} 2 \left(\varepsilon_u l_{deb} + \frac{1}{E_2}\left(\left(f_u - \zeta + \frac{4\tau_y}{ø_s} l_{deb}\right)(½s_{rm} - l_{deb}) - \frac{2\tau_y}{ø_s}\left(\frac{1}{4}{s_{rm}}^2 - {l_{deb}}^2\right)\right)\right) \tag{16b}$$

In Figure 5b, an example of the variation of the reinforcement strains is shown, in which the meaning of $\varepsilon_{average}$ is also illustrated. In Figure 5a, the bold, fully drawn line shows the modeled stress-strain curve of an embedded reinforcement bar. It is this stress-strain curve that is to be applied in the following. By refereeing to the bold line in Figure 5a, the corresponding reinforcement stresses can be determined as:

$$\sigma_s = \begin{cases} \varepsilon_s E_s & \text{for } \varepsilon_s \leq \varepsilon_y \\ \zeta + \varepsilon_s E_2 & \text{for } \varepsilon_s \geq \varepsilon_y \end{cases} \tag{17}$$

What regards the strains in the concrete, only knowledge of the principal (compressive) strain, at an angle θ with respect to the *x*-axis, is available. The principal concrete strain perpendicular to this compressive strain is not known. It is known that it is a positive strain, but the magnitude is unknown. Accordingly, the size of the strain in the concrete is also unknown in other sections a priori. The average strain in the disk in the *x*-*y* direction is identical to the average strain in the reinforcement in these directions. Hence, the strains in the disk are known in the directions of the *x* and *y* axis. Knowledge of these three strains is sufficient to determine the displacements at any point in the disk, as well as on the boundaries.

2.3 Displacements on the boundaries

The potential energy also includes the contribution from the external stresses, whereby knowledge of the displacements on the boundaries are necessary. On basis of the previously mentioned known strains, an expression for the boundary displacements can be derived. For a disk, or piece of a disk, with in-plane dimensions *h·b*, the displacements in points F and G, can be found (see appendix A)

Point F: $u(x,y) = u(0, ½h) = \left(\frac{1}{2}\frac{\varepsilon_c + \varepsilon_{sx}}{\tan\theta}\cdot h, \frac{1}{2}\varepsilon_{sy}\cdot h\right)$ (18)

Point G: $u(x,y) = u(½b, 0) = \left(\frac{1}{2}\varepsilon_{sx}\cdot b, \frac{1}{2}(\varepsilon_c + \varepsilon_{sy})\tan\theta \cdot b\right)$ (19)

Using these expressions, the loss in potential energy owing to the displacement of external forces can be determined.

2.4 The corrected potential energy

Each of the three parts included in the corrected potential energy is illustrated on Figure 6. It is assumed that yielding in the reinforcement in the *y*-direction is initiated first. Hence, the reinforcement in the direction of the x-axis as well as the concrete remains in the elastic range.

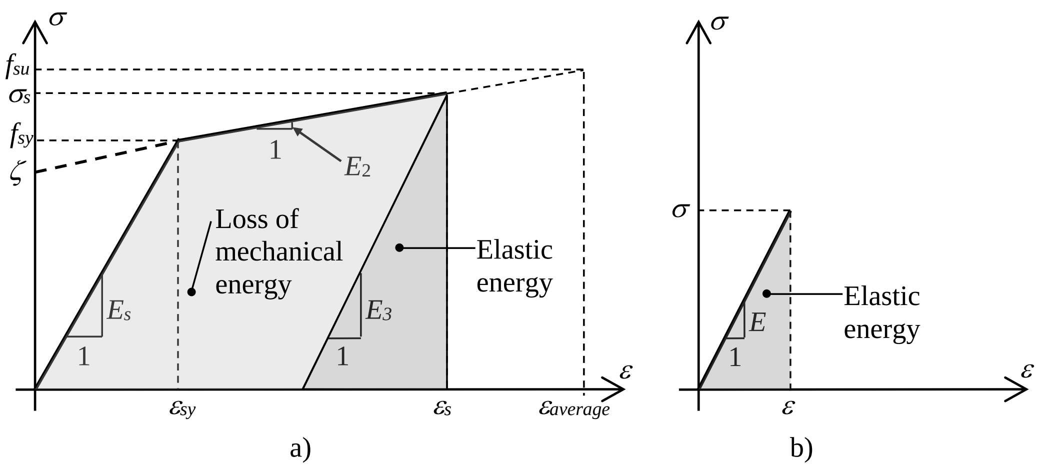

Figure 6 Illustration of elastic energy and lost mechanical energy

2.4.1 The elastic energy

The elastic energy per unit volume of the disk is given by:

$$\Pi_{elastic} = \frac{1}{2}\sigma_c\varepsilon_c + \frac{1}{2}\sigma_{sx}\varepsilon_{sx}\rho_x + \frac{1}{2}\frac{{\sigma_{sy}}^2}{E_3}\rho_y \quad (20)$$

2.5.2 The loss of potential energy from the external stresses

Based on expressions (18) an (19) the loss of potential energy per unit of volume ($V = h \cdot b \cdot t$) owing to the external forces (*i.e,* the resultants of τ_{xy}, σ_x and σ_y) can be expressed as:

$$\Pi_P = \frac{1}{V}\cdot 2\begin{pmatrix} \sigma_x \cdot t \cdot h \cdot \frac{1}{2}\varepsilon_{sx} \cdot b + \sigma_y \cdot t \cdot b \cdot \frac{1}{2}\varepsilon_{sy} \cdot h + \\ \tau_{xy} \cdot t \cdot b \cdot \frac{1}{2}\frac{\varepsilon_c + \varepsilon_{sx}}{\tan\theta} \cdot h + \tau_{xy} \cdot t \cdot h \cdot \frac{1}{2}(\varepsilon_c + \varepsilon_{sy})\tan\theta \cdot b \end{pmatrix}$$

$$\Pi_P = \sigma_x\varepsilon_{sx} + \sigma_y\varepsilon_{sy} + \tau_{xy}\varepsilon_c\left(\frac{1}{\tan\theta} + \tan\theta\right) + \tau_{xy}\frac{\varepsilon_{sx}}{\tan\theta} + \tau_{xy}\varepsilon_{sy}\tan\theta$$

Expression (2) - (4) are rewritten and inserted:

$$|\tau_{xy}| = \sigma_c sin\theta cos\theta$$

$$\frac{|\tau_{xy}|}{\tan\theta} = \sigma_{sx}\rho_x - \sigma_x$$

$$|\tau_{xy}|\tan\theta = \sigma_{sy}\rho_y - \sigma_y$$

$$\Pi_P = \sigma_x\varepsilon_{sx} + \sigma_y\varepsilon_{sy} + \sigma_c\varepsilon_c + (\sigma_{sx}\rho_x - \sigma_x)\varepsilon_{sx} + (\sigma_{sy}\rho_y - \sigma_y)\varepsilon_{sy}$$

$$\Pi_P = \sigma_c\varepsilon_c + \sigma_{sx}\varepsilon_{sx}\rho_x + \sigma_{sy}\varepsilon_{sy}\rho_y \quad (21)$$

2.4.3 The loss of potential energy due to non-elastic material behavior

As mentioned, it is assumed that yielding is initiated in the reinforcement in the *y*-direction for a load lower than what is required for yielding the in the x-direction to develop. The loss of potential energy due to the development of plastic strains therefore stems only from the reinforcement in the direction of the *y*-axis.

In accordance with the notation on Figure 6a, the loss of potential energy pr. unit volume can then be determined as,

$$\Delta w = \sigma_{sy}\varepsilon_{sy} - \frac{1}{2}\zeta\varepsilon_y - \frac{1}{2}(\sigma_{sy} - \zeta)\varepsilon_{sy} - \frac{1}{2}\frac{{\sigma_{sy}}^2}{E_3} \quad (22)$$

where ε_{sy} is determined as, see formula (17):

$$\varepsilon_{sy} = \frac{\sigma_{sy} - \zeta}{E_2}$$

Inserted into formula (22):

$$\Delta w = \sigma_{sy}\varepsilon_{sy} - \frac{1}{2}\zeta\varepsilon_y - \frac{1}{2}\frac{(\sigma_{sy} - \zeta)^2}{E_2} - \frac{1}{2}\frac{{\sigma_{sy}}^2}{E_3} \quad (23)$$

2.4.4 Expressing the corrected potential energy

By inserting expressions (20), (21) and (23) into (5), and by expressing the strain by stresses and stiffness`, see (14) and (17) the corrected potential energy per volume unit is expressed by:

$$\Pi = -\frac{1}{2}\left(\zeta \frac{f_y}{E_s}\rho_y + \frac{{\sigma_c}^2}{E_c} + \frac{{\sigma_{sx}}^2}{E_s}\rho_x + \frac{\left(\sigma_{sy} - \zeta\right)^2}{E_2}\rho_y\right)$$

By using expressions (2) - (4), σ_c, σ_{sx} and σ_{sy} are eliminated from the expression, as these are expressed by the external stresses σ_x, σ_y and τ_{xy}, as well as the orientation of the inclined concrete compression θ and the degree of reinforcement ρ_x and ρ_y. This yields the following expression of the corrected potential energy.

$$\Pi = -\frac{1}{2}\left(\frac{1}{E_s}\zeta f_y \rho_y + \frac{1}{E_c}{\tau_{xy}}^2\left(\tan\theta + \frac{1}{\tan\theta}\right)^2 + \frac{1}{\rho_x E_s}\left(\sigma_x + \frac{\tau_{xy}}{\tan\theta}\right)^2 + \frac{1}{\rho_y E_2}\left(\sigma_x + \tau_{xy}\tan\theta - \zeta\rho_y\right)^2\right) \qquad (24a)$$

By replacing E_2 with E_s in (24a) and consequently $\zeta = 0$, the potential energy for the purely linear elastic case is obtained.

If the concrete strains are larger than the proportionality limit, ε_{co}, the part containing the contribution from the concrete will change, whereby the corrected potential energy is to be determined as;

$$\Pi = -\frac{1}{2}\left(\frac{1}{E_s}\zeta f_y \rho_y + \nu f_c \varepsilon_{co} + \frac{1}{\rho_x E_s}\left(\sigma_x + \frac{\tau_{xy}}{\tan\theta}\right)^2 + \frac{1}{\rho_y E_2}\left(\sigma_x + \tau_{xy}\tan\theta - \zeta\rho_y\right)^2\right) \qquad (24b)$$

2.5 Equilibrium

The state of equilibrium is finally derived by applying the expression in (24a) or (24b) in combination with the condition stated in (6). Assuming the concrete to behave linearly elastic ($\varepsilon_c \leq \varepsilon_{co}$) the following equation for $\tan\theta$ is thereby obtained;

$$(\tan^4\theta - 1) - \frac{E_c}{E_s\rho_x}\left(\frac{\sigma_x}{\tau_{xy}}\tan\theta + 1\right) - \frac{E_c}{E_2\rho_y}\left(\tan^4\theta - \left(\frac{\rho_y\zeta - \sigma_y}{\tau_{xy}}\right)\tan^3\theta\right) = 0 \text{ for } \varepsilon_{sx} \leq \varepsilon_y, \varepsilon_{sy} \geq \varepsilon_y \qquad (25a)$$

The corresponding value of the principal stress direction θ ensures that the geometric and static conditions are both fulfilled. For the case where $\varepsilon_c \geq \varepsilon_{co}$ the equation for $\tan\theta$ changes into;

$$-\frac{E_c}{E_s\rho_x}\left(\frac{\sigma_x}{\tau_{xy}}\tan\theta + 1\right) - \frac{E_c}{E_2\rho_y}\left(\tan^4\theta - \left(\frac{\rho_y\zeta - \sigma_y}{\tau_{xy}}\right)\tan^3\theta\right) = 0 \text{ for } \varepsilon_{sx} \leq \varepsilon_y, \varepsilon_{sy} \geq \varepsilon_y \qquad (25b)$$

2.6 Required strain capacity

The shear capacity of the disk is determined according to the expression in (1) for known stresses σ_x and σ_y. Having determined the principal concrete stress direction θ using (25a) or (25b) the stress in the reinforcement in the *y*-axis direction is determined as;

$$\sigma_{sy} = \frac{\sigma_y}{\rho_y} + \frac{\tau_{xy}\tan\theta}{\rho_y} \tag{26}$$

The corresponding strain is obtain by inserting (26) into (10)

$$\varepsilon_{sy} = \frac{\sigma_y + \tau_{xy}\tan\theta - \zeta\rho_y}{E_2\rho_y} \tag{27}$$

The hereby determined strain is the largest strain in the reinforcement. The largest determined strain for a given combination of σ_x, σ_y, and τ_{xy} must be lower or equal to the strain capacity.

2.7 The dependency of the concrete strain on the strain capacity

The required strain capacity is determined on the basis of the expression in (24a) which assumes a linear elastic behavior of concrete, or (24b) when $\sigma_c = \nu fc$. With a plastic behavior of the concrete ($\sigma_c = \nu fc$), the corrected potential energy in the concrete becomes independent of the inclination of the compression in contrast to an elastic behavior where the corrected potential energy increases as a function of decreasing inclination (relative to 45^o). At the transition to a plastic behavior of the concrete, the inclination of the compressive stress drops slightly towards the direction of the strongest reinforcement. The strain in the direction of the weakest reinforcement is thereby also reduced and, consequently, the required strain capacity decreases. The most critical scenario in regards to the required strain capacity thus generally occurs for a scenario where the concrete behaves elastic ($\sigma_c < \nu fc$) which is often the case for disks in practice.

For known materials (f_y, f_u, E_s, ε_u, f_c and ε_{co}) and reinforcement degrees (ρ_x and ρ_y) the available strain capacity is determined by the expression in (13), by considering the direction with the smallest degree of reinforcement. The orientation of the concrete compressive stresses (θ) is determined by formula (25a) for any combination of σ_x, σ_y, and τ_{xy} fulfilling formula (9).

3. Comparison with tests

The model has been compared with selected tests performed and reported by Vecchio and Collins [1982]. The comparison involves the tests with large anisotropic reinforcement layouts only.

With respect to the ultimate stress capacity f_u, and the corresponding strain for the bar reinforcement these are generally not available in [1982]. One stress-strain curve is shown in the report for one coupon from PV09. This curve has in this comparison been generalized, so f_u/f_y = 1.1 and ε_{su} = 13% has been used in the comparison. Due to the uncertainties on these values also the more extreme curves with f_u/f_y = 1.1±0.05 and ε_{su} = 13% ± 6.5% are sketched with dotted lines. The dependencies on these values is seen to be limited on the variation of θ, but do play and important role on the maximum strain in the reinforcement at specific load levels.

In Figure 7 is seen comparisons on the variation of the measured and modeled orientation of the inclined stress in the concrete on PV18 and PV19.

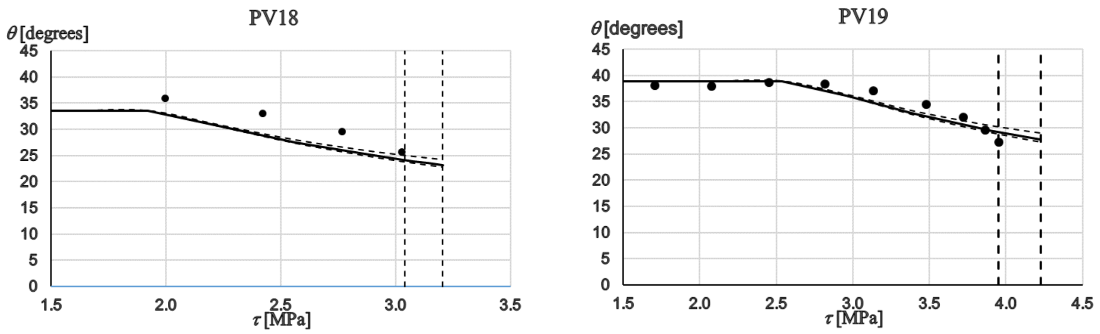

Figure 7 Comparisons on the variation of the measured and modeled orientation of the inclined compression in PV18 and PV19

In Table 2 is shown the calculated maximum strain in the transverse reinforcement (given as $\varepsilon_{s,max}/\varepsilon_{su}$) at maximum load.

Table 2.

	$\varepsilon_{s,max}/\varepsilon_{su}$		
	f_u/f_y = 1.10, ε_{su} = 13%	f_u/f_y = 1.15, ε_{su} = 6.5%	f_u/f_y = 1.05, ε_{su} = 20.5%
PV18	45%	62%	40%
PV19	19%	30%	16%

It is seen from Table 2, that there is a relative large deviation on the $\varepsilon_{s,max}/\varepsilon_{su}$ as function of the estimated variations of f_u/f_y and ε_{su}. However, none of the estimated variations led to rupture of the reinforcement. This was also not observed in the experiments.

In Table 3 is shown a comparison of a total of 7 tests with anisotropic reinforcement performed by Vecchio and Collins [1982].

Table 3. At maximum load

	PV10	PV12	PV18	PV19	PV20	PV21	PV29
θ_{model}	36.6	22.9	23.2	27.4	30.2	34.9	40.3
θ_{test}	34.5	24.9	25.7	27.3	30.3	32.4	35.4
$\tau_{u,model}$	3.70	3.22	3.21	4.23	4.71	5.73	5.94
$\tau_{u,test}$	3.97	3.13	3.04	3.94	4.26	5.03	5.87

The modeled value of both θ and τ are seen to fit very well with the test results.

4. Maximum anisotropic ratio.

The maximum permissible difference between reinforcement in the *x*- and *y*-direction depends on the degree of reinforcement and of concrete strength. In the following, the maximum permissible difference, represented as $(\rho_x/\rho_y) = (\rho_{max}/\rho_{min})$ as function of the lowest degree of reinforcement for three different concrete strengths (f_c = 30 MPa, f_c = 60 MPa, f_c = 90 MPa) for class A, B and C according to EC2 [2008].

In Figure 8, the strain capacity of the embedded reinforcement itself is determined by formula (13). The strain capacity is in addition to reinforcement class and concrete reinforcement a function of the degree of reinforcement.

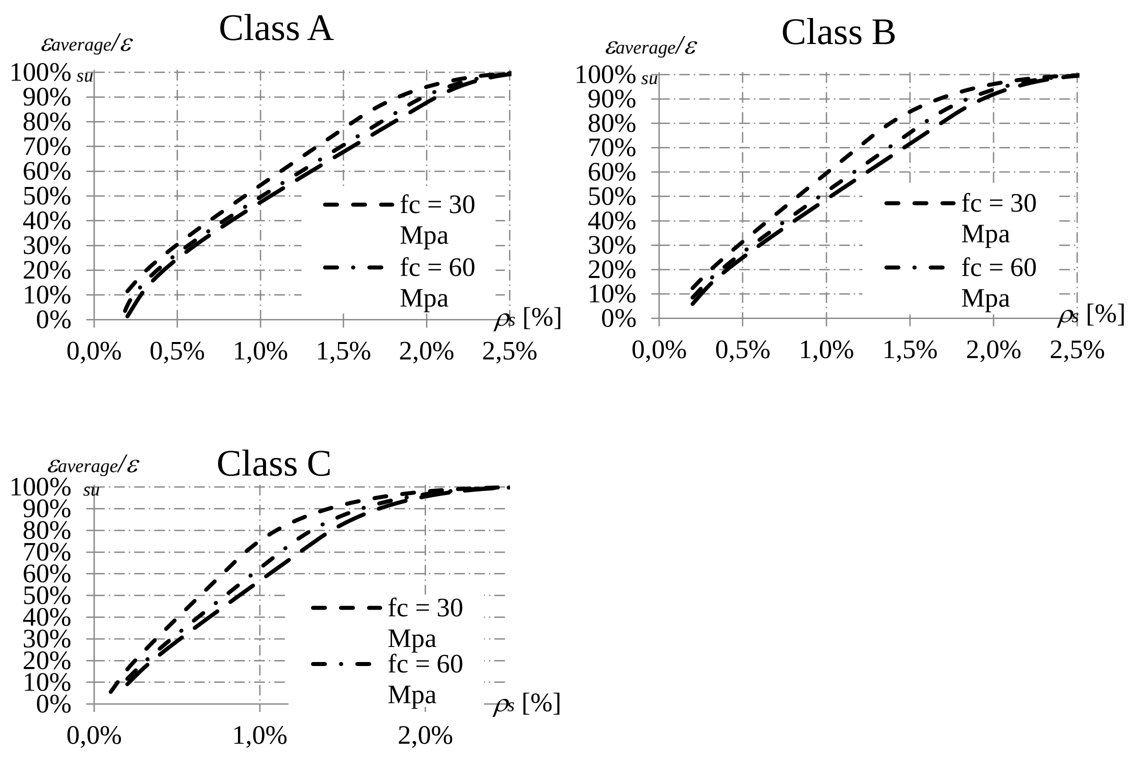

Figure 8 Strain capacity of the embedded reinforcement itself

Table 4 shows the data used to determine $\varepsilon_{average}$.

Table 4

Class	f_y	f_u	E_s	ε_{su}	f_c	A_c
	[MPa]	[MPa]	[MPa]	[%]	[MPa]	[mm^2]
A	500	525	200,000	2.5	30;60;90	100 x 100
B	500	540	200,000	5.0	30;60;90	100 x 100
C	500	575	200,000	7.5	30;60;90	100 x 100

In Figure 9 is shown the maximum values of the ratio between maximum and minimum degree of reinforcement in the two directions, ensuring that the strain capacity of the reinforcement is sufficient for any combination of σ_x, σ_y and τ_{xy} in order to reach the strength given by the yield condition.

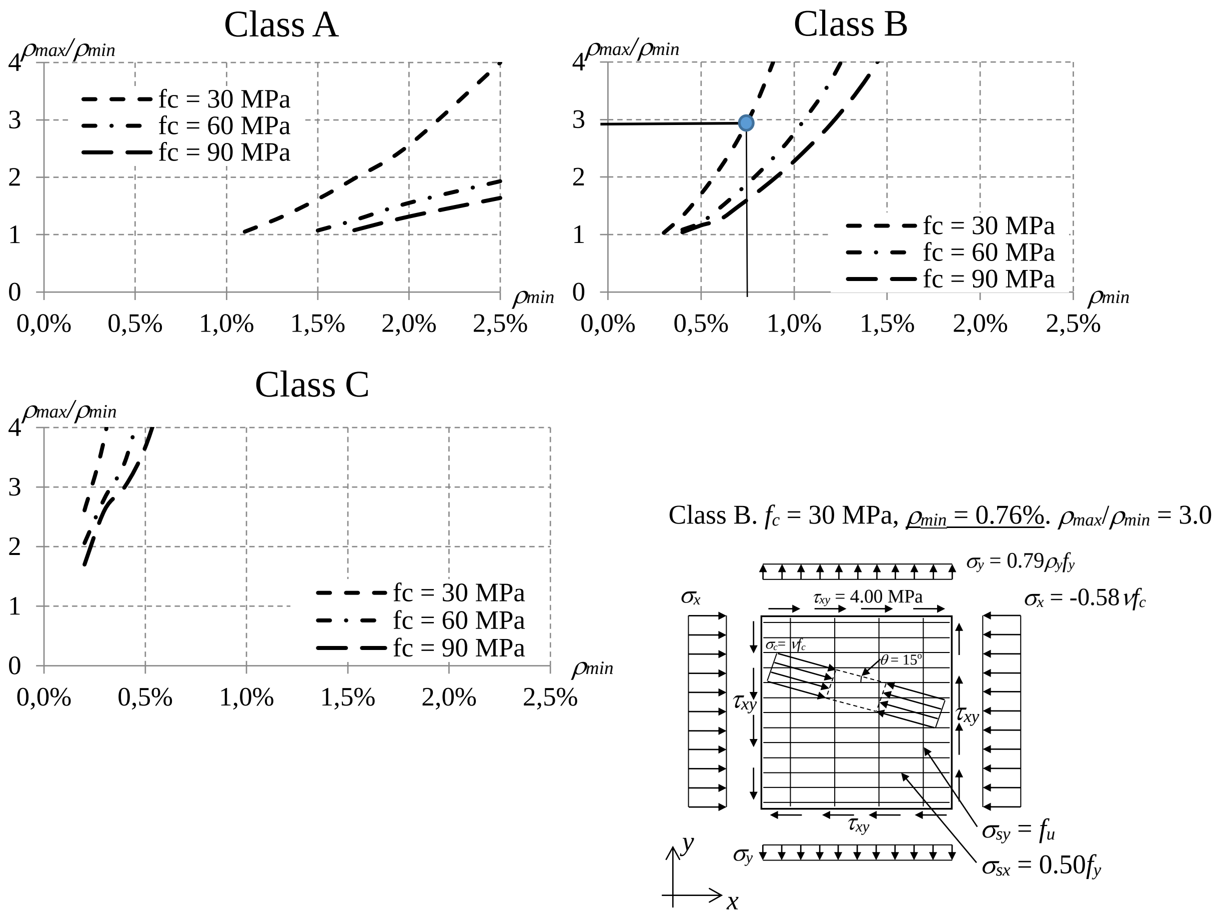

Figure 9 Maximum values of the ratio between maximum and minimum degree of reinforcement, ensuring that the strain capacity of the reinforcement is sufficient

As an example of how to read Figure 9 a minimum degree of reinforcement of ρ_{min} = 0.76% requires that ρ_{max}/ρ_{min} do not exceed 3.0 for a concrete with f_c = 30 MPa, in the case where the reinforcement can be classified as Class B. For ρ_{min} = 0.76% the strain capacity according to (27) is 46% of the strain capacity of the bar reinforcement, i.e. 0.46 x 5% = 2.3%, see also Figure 8. The combination of τ_{xy}, σ_x and σ_y leading to the largest possible strain in the reinforcement is seen in Figure 9.

By reading in Figure 9b (Class B), a minimum degree of reinforcement of 0.3% (f_c = 30 MPa) and 0.4% (for f_c = 60 MPa and f_c = 90 MPa), respectively, is required in order to have sufficient strain capacity for the disk in the isotropic case to achieve the calculated capacity according to (9). Figure 4 indicates that the calculated crack distances for small degrees of reinforcement are significantly higher than the measured crack distances. The crack distance is a very important factor in determining $\varepsilon_{average}$.

Figure 10a shows a diagram equivalent to Figure 9b, with the modification that the external stress in the direction with the largest degree of reinforcement is kept equal to zero. It is seen that the curves are changed significantly. For instance it is seen that in order to reach ρ_{max}/ρ_{min} = 3.0 there is only required a minimum degree of reinforcement of ρ_{min} = 0.48% for class B reinforcement.

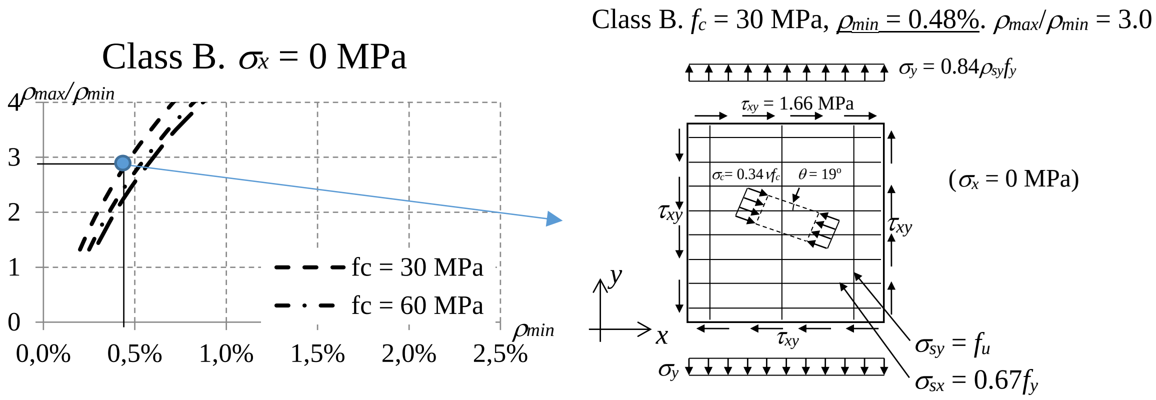

Figure 10 Maximum values of the ratio between maximum and minimum degree of reinforcement, ensuring that the strain capacity of the reinforcement is sufficient in the case of no compression on the disk

5. Conclusion

The method used is based on yield condition for disks. This is developed under the assumption of ideal plastic material behavior. Due to the ideal plastic material modeling, it can not be verified whether the reinforcement has sufficient strain capacity to reach the capacity determined by the yield condition.

The method which is presented in this paper can be used to determine and ensure that the required strain capacity for a given design and for a given load combination is present.

For reinforcement of class A, B and C according to EC2, limits have been set up in Figure 9 for maximum anisotropic ratio as function of minimum reinforcement and concrete strength. The limits will ensure that the strain capacity is sufficient to reach the specific capacity regardless of the combination of external stresses.

References

Nielsen, M. P. (1969) Om jernbetonskivers styrke, Copenhagen, Polyteknisk forlag, 1969, 254 pp. (English edition: On the strength of reinforced disks, Acta Polytech. Scand., Copenhagen 1970, 261 pp.)

Vecchio, F. and Collins, M. P. (1982) The Response of Reinforced Concrete to in-plane shear and Normal Stresses, The Department of Civil Engineering, University of Toronto, Canada, Mar. 1982, 332 pp.

Hagsten, L. G. (2017) Equilibrium of Truss and Beam Structures of Inelastic Materials, Journal of Civil Engineering and Management, Vol. 23, Issue 5, 2017.

Ruiz M. Fernandéz, A. Muttoni, P. Gambarova. (2007) Analytical modelling of the pre- and post-yield behavior of bond in reinforced concrete, ASCE Journal of Structural Engineering, Vol. 133, No 10, Reston, USA, 2007, pp. 1364-1372.

Lee, S. C.; Cho, J. Y.; Vecchio, F. J. (2011). Model for post-yield tension stiffening and rebar rupture in concrete members. Engineering Structures 33 (5), 1723-1733.

Y Goto. (1971) Cracks Formed in Concrete Around Deformed Tension Bars, ACI Journal, Volume 68, No. 4, April 1971, pp 244-251.

P. Marti, M. Alvarez, W. Kaufmann, V. Sigrist. (1998) Tension Chord Model for Structural Concrete, Structural Engineering International, IABSE, Volume 8, Number 4, 1 November 1998, pp. 287-298.

J. Jokela. (1986) Dimensioning of Strain or Deformation – Controlled Reinforced Concrete Beams. Technical Research Centre of Finland. Publication 33. 1986.

EN-1992-1-1 (2008), Eurocode 2 – Design of Concrete Structures. Part 1–1: General Rules and Rules for Buildings, CEN, 2008.Bailey and Moore (2000) Electrochemical removal of chlorides from concrete. In Proceedings of a Conference on the Rehabilitation of Concrete Structures (Smith DW and Lewis F (eds)). Thomas Telford, London, UK, pp. 2–30.

Appendix A Displacements at the boundary

There is considered a rectangular section with side lengths l_x and l_y. The side lengths are adjusted such that the diagonal forms the angle θ with the x-axis. Origo is located in the center of the section. Figure A1 outlines the consequence for the strains made by the prerequisites.

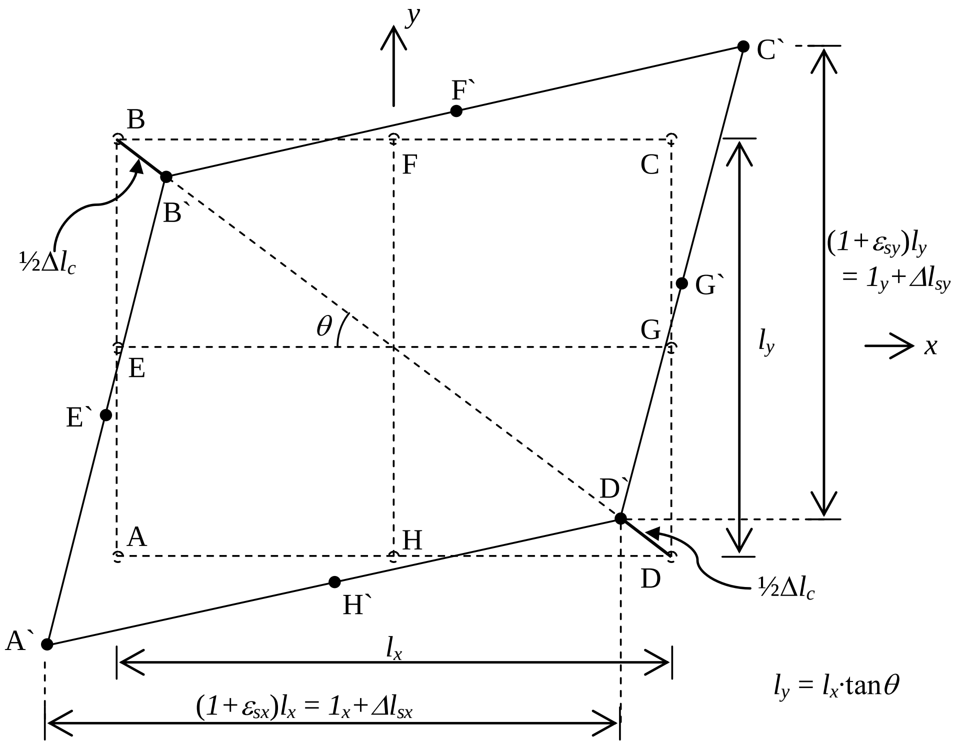

Figure A1 Known deformation in the disk

The displacements of the points A-D are as shown in Figure A2. As a consequence of the displacements of A-D, the displacements of the centers of the individual sides, points E-H, can be determined.

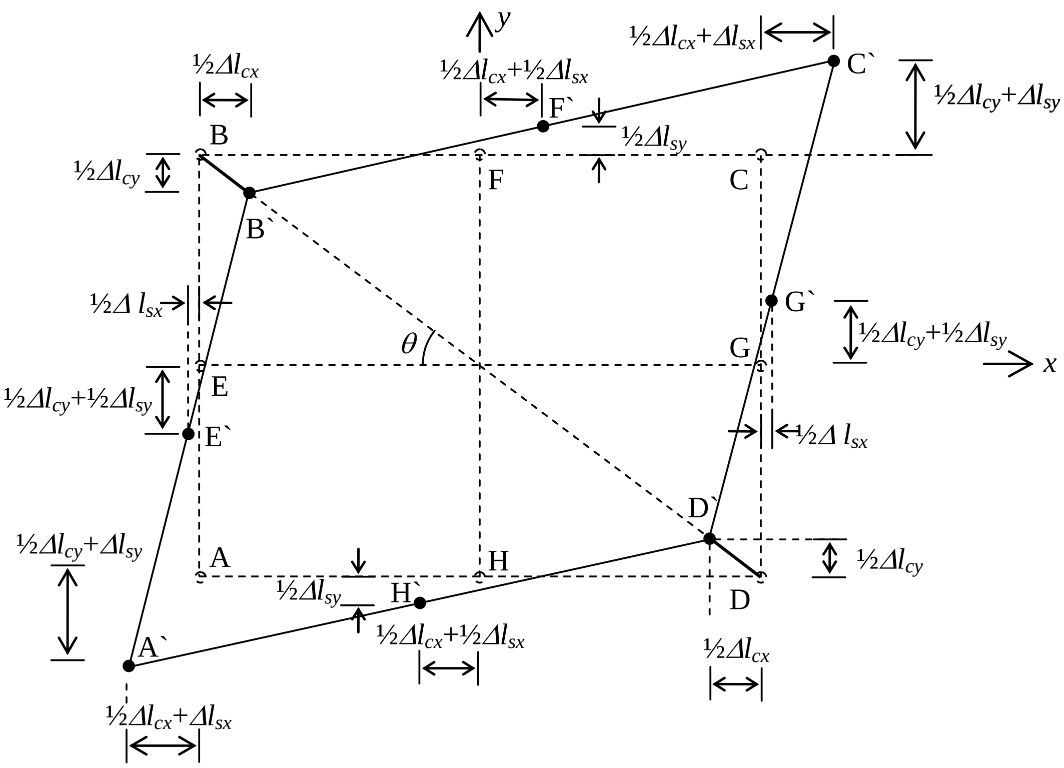

Figure A2 Definition of geometrical variables

The displacements in the direction of the *x*-axis can be written:

$$\left(u_{x,x}, u_{y,x}\right) = \left(\frac{½\Delta l_{sx}}{½l_x}x, \frac{½\Delta l_{cy} + ½\Delta l_{sy}}{½l_x}x\right)$$

The displacements in the direction of the *y*-axis can be written:

$$\left(u_{x,y}, u_{y,y}\right) = \left(\frac{½\Delta l_{cx} + ½\Delta l_{sx}}{½l_y} + \frac{½\Delta l_{sy}}{½l_y}y\right)$$

The displacements in an arbitrary point can therefor be written:

$$\left(u_x, u_y\right) = \left(\frac{½\Delta l_{sx}}{½l_x}x + \frac{½\Delta l_{cx} + ½\Delta l_{sx}}{½l_y}, \frac{½\Delta l_{cy} + ½\Delta l_{sy}}{½l_x}x + \frac{½\Delta l_{sy}}{½l_y}y\right)$$

The following relationships can be established between the strains and the displacements:

$$\Delta l_c = \varepsilon_c l_{diag} = \varepsilon_c \frac{l_x}{\cos\theta}$$

$$\Delta l_{cx} = \varepsilon_c \frac{l_x}{\cos\theta}\cos\theta = \varepsilon_c l_x,\ \Delta l_{cy} = \varepsilon_c \frac{l_x}{\cos\theta}\sin\theta = \varepsilon_c l_x \tan\theta$$

$$\Delta l_{sx} = \varepsilon_{sx} l_x,\ \Delta l_{sy} = \varepsilon_{sy} l_y = \varepsilon_{sy} l_x \tan\theta$$

Inserted:

$$\left(u_x, u_y\right) = \left(\varepsilon_{sx}x + \frac{\varepsilon_c + \varepsilon_{sx}}{\tan\theta}y, \left(\varepsilon_c + \varepsilon_{sy}\right)\tan\theta\ x + \varepsilon_{sy}y\right) \quad \text{(A1)}$$

For a disk or a piece of a disk with the dimension $h \cdot b$ in the plane, the displacements in points F and G, see Figure A3 can be determined to:

$$\text{Point F: } u(x,y) = u(0, ½h) = \left(\frac{1}{2}\frac{\varepsilon_c + \varepsilon_{sx}}{\tan\theta} \cdot h, \frac{1}{2}\varepsilon_{sy} \cdot h\right) \quad \text{(A2)}$$

$$\text{Point G: } u(x,y) = u(½b, 0) = \left(\frac{1}{2}\varepsilon_{sx} \cdot b, \frac{1}{2}\left(\varepsilon_c + \varepsilon_{sy}\right)\tan\theta \cdot b\right) \quad \text{(A3)}$$

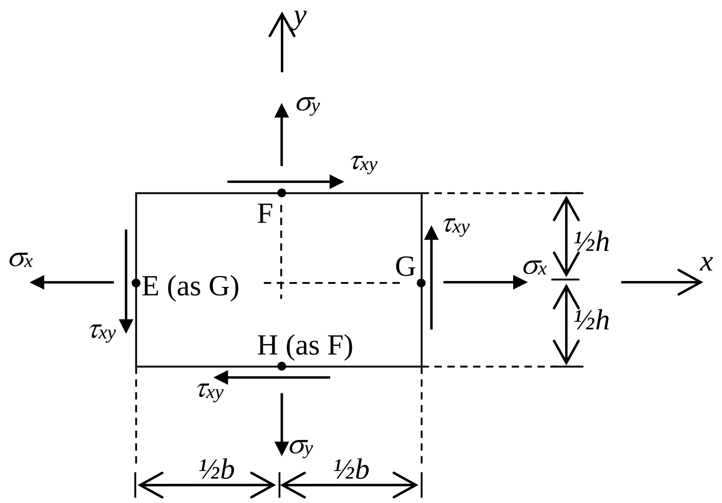

Figure A3 Deformation on the edge of the disk

List of notation

b	Width of disk
E_s	Youngs modulus for reinforcement
E_2	Strain hardening modulus for reinforcement
E_3	Youngs modulus at unloading after yielding
E_c	Youngs modulus for concrete
f_y	Yield strength of reinforcement
f_u	Ultimate strength of reinforcement
f_c	Uniaxial compressive strength of concrete
f_{ck}	Characteristic compressive strength of concrete
f_{cm}	Average compressive strength of concrete
h	Height of disk
l_{deb}	Extension of debonded zone at cracks
l_p	Length of zone at cracks at with yielding
s_{rm}	Distance between cracks
t	Depth of disk
V	Volume of disk
Δw	Loss of mechanical energy

$ø_s$	Diameter of reinforcement bar
ε_c	Strain in concrete
ε_{co}	Max strain in concrete for elastic behavior
$\varepsilon_{average}$	Average strain in embedded reinforcement
$\varepsilon_{capacity}$	Average strain in embedded reinforcement
ε_x	Strain in direction of *x*-axis
ε_{sx}	Strain in reinforcement indirection of *x*-axis
ε_y	Strain in direction of *y*-axis
ε_{sy}	Strain in reinforcement indirection of *y*-axis
ε_{su}	Ultimate strain in bar reinforcement
Π	Corrected potential energy
$\Pi_{elastic}$	Elastic potential energy
Π_P	Potential energy from external load
θ	Orientation of inclined concrete stress
ν	Effectiveness factor
ζ	Virtual stress (for $\sigma_s = 0$) for strain hardening part
ρ_s	Degree of reinforcement
ρ_x	Degree of reinforcement in direction of *x*-axis
ρ_y	Degree of reinforcement in direction of *y*-axis
σ_c	Inclined, principal stress in concrete
σ_s	Stress in reinforcement
σ_x	External stress in direction of *x*-axis
σ_{cx}	Stress in concrete in direction of *x*-axis
σ_{sx}	Stress in reinforcement in direction of *x*-axis
σ_y	External stress in direction of *y*-axis
σ_{cy}	Stress in concrete in direction of *y*-axis
σ_{sy}	Stress in reinforcement in direction of *y*-axis
τ_e	Elastic shear between reinforcement and concrete
τ_y	Plastic shear between reinforcement and concrete
τ_{xy}	External shear stress
τ_{cxy}	Shear stress on concrete
τ_u	Ultimate shear capacity

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