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A Model Study of Column Stability based on Residual Stress Theory

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A Model Study of Column Stability based on Residual Stress Theory

Thomas Kabel[1]
Mikkel Kongsmark Andersen[1]
Lars German Hagsten[2]

Abstract

The present article is based on a master's thesis project on the study of steel columns buckling resistance. The primary focus is on the non-linear buckling region, where both imperfections and residual stresses influence the buckling strength of a column. In the current European Standard, this is carried out by applying the Ayrton-Perry model. In this model, residual stresses are equated as geometrical imperfections. In addition, a model proposed by Lehigh University is considered. This model expresses the buckling strength of a column based on the residual stress distributions influence on the gradually yielding of its cross-section. In order to optimize the Lehigh model, a modification which include the influence of geometrical imperfections is proposed. Furthermore, the Lehigh model and the Ayrton-Perry model are combined, in an effort to investigate their combined influence on the buckling strength of a column. Additionally, the article investigates the basis for establishing a model based on energy principles, in which the influence of residual stresses are accounted for. It is concluded that the modified Lehigh model reduce the buckling strength in relation to the original model, however not as much as the Ayrton-Perry model. Generally, the article draws the conclusion that it is highly troublesome to establish a simple model, in which the ac-

[1] M.Sc., Aarhus University, Aarhus, Denmark
[2] Professor (Ingeniørdocent), Aarhus University, Aarhus, Denmark

tual influence of residual stresses are included. This is primarily due to the erratic behaviour of residual stresses.

1. Introduction

An extensive and increasingly developing discipline in the field of structural engineering, is the optimization of structural designs. As noted by Salmon *et al*. [20], the optimization is based on predetermined criteria such as cost-efficiency, construction time, weight, etc. Consequently, these criteria often involve an optimization of the structural systems performance. Naturally, this requires a profound knowledge of the systems behaviour and the properties of the material, in order not to compromise the integrity of the system. A noteworthy example of this is the stability of columns. If an ideal linear-elastic column is considered, it fails either by buckling or material failure as illustrated in fig. 1.1. Here, the hatched area depicts how buckling severely reduces the material utilization. Thus, an optimized design of the column system will ensure that the slenderness ratio of the column falls within the highest material utilization. However, this warrants for a high knowledge of the behaviour of the column system in this specific range, as it assumes a full utilization of the material. The focus of this article is to investigate this range of a columns capacity.

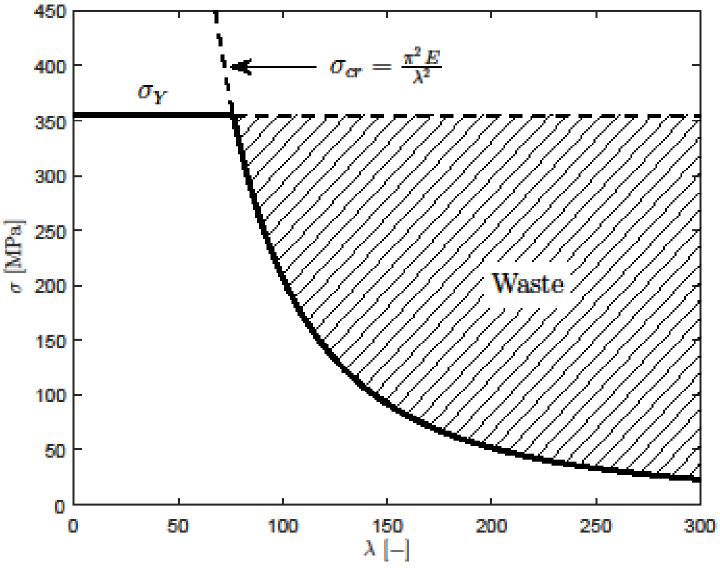

Figure 1.0.1 Column stability diagram along with the waste area created by Eulers formula

1.1 Historical review

In 1744 Euler published his famous column formula for the critical load of an ideal built-in/free column, with $2l$ as the column length, subjected to a concentric axial force

$$P_{cr} = \frac{\pi EI}{(2l)^2} \tag{1}$$

At the time, the flexural stiffness, EI, was an unknown quantity. As seen in Timoshenko [21], Euler instead denoted this as a constant C, which he referred to as the "absolute elasticity" of the body. Euler had also by the aid of Bernoulli, expressed the relation between the deflection of a built-in/free beam and transverse load at the free end. He did so by the principle of minimum potential energy of a system, and variational calculus. This enabled Euler to experimentally determine C and ultimately to determine a columns buckling strength.

However, Euler's formula was given little attention as it severely overestimated axially compressed structures buckling resistance. It was not until the late 19th century that Engesser and Consideré realized that the column was not ideal, as Euler assumed. Through experiments they showed independently[3] that the column becomes inelastic prior to buckling Thus, in 1895 Engesser published his so-called "tangent-modulus theory", in which he assumes that inelastic buckling occurs with no increase in load. His method states that in the inelastic region of the column's response, a variable elastic modulus $E_t = \mathrm{d}\sigma / \mathrm{d}\varepsilon$ should be applied instead of Euler's constant value in (1). Comparing this method with experimental data showed satisfactory results, although higher strengths were observed than those calculated using his method. As Consideré worked independently on similar topics, he discovered an erroneous neglect in Engessers model. As the column experiences flexure at the critical load, the stress distribution in the column changes due to additional bending. As such, at the convex surface of the flexed column, tensile stress would superimpose and create strain reversal. Thus, a partial part of the cross-section would regain flexural stiffness. Consequently, Engesser revised his theory with the so-called "double-modulus theory" in which he introduced the required strain reversal contribution through an additional modulus. However, according to Salmon *et al.* [20], this approach yielded higher critical load values than those obtained experimentally. It was not until 1946, when Shanley published his work, it became evident that the reduced or double modulus could not be reached. As stated in Salmon *et al.* [20], Shanley realized that as each increment of curvatures is obtained, the value of the loaded force would increase as long as the increase in compressive force would be larger than the increase in tension force, when looking at equilibrium. The double modulus theory did not accept this assumption and considered only equilibrium positions near the perfectly straight one. Therefore, the load at the tangent modulus, P_t, is a lower bound solution and the reduced modulus load, P_r, is an upper bound solution which it will never reach. Simultaneously with Shanley's solution to the column behaviour, Lehigh University commenced an extensive investigation of the causes for

[3] Ibid.

the non-linear behaviour observed in steel columns. Among others, a sponsorship from the Column Research Council, the Pennsylvania Department of Highways and the Bureau of Public Roads enabled the possibility of the extensive study. A fundamental result of this study was that residual stress was the main reason for the non-linear behaviour and not the influence of unavoidable imperfections, as was the popular belief at the time. The first published work from Lehigh, in which the non-linear behaviour of a column was explained by the presence of residual stress, was in 1941[4]. This was later discussed and tested in Luxion and Johnston [19]. As part of the research program, Adolf Huber conducted a three year theoretical and experimental study, which were rendered into a dissertation, see Huber and Beedle [12]. The dissertation, along with the pilot investigation published in Beedle and Huber [2], is the foundation for the Lehigh model presented in this article.

1.2 Current theory

The contemporary approach in the design of columns, as proposed by the European Standard, DS/EN-1993 [4], is based on experimental research as well as theoretical, numerical and probabilistic investigations performed in the 1960-1970's. However, due to the incorporation of the earlier theoretical work by Ayrton and Perry, which included the influence of imperfections, the model was revised in the period from 1980-1990. It was also in this period that the five column buckling curves (a_0,a,b,c,d) were introduced through tabulated values, see ECCS [6]. These curves are dependent on whether buckling will be susceptible to occur about the major or minor axis. Further, they are also dependent on the geometry of the cross-section, which is in correlation with the magnitude and distribution of residual stresses for different cross-sectional types. From 1992-2005, the European Standard became unified, resulting in the tabulated values for the buckling curves being represented by actual curves. This representation is still used in DS/EN-1993 [4]. As a part of the overall model study in this article, the Ayrton-Perry model is obtained and employed as the foundation on which the other models should be compared. This is due to the model being well-based both theoretical and experimentally, Alpsten [1]. The issues regarding the current model is however that the basis for the calculations is a linear distribution of residual stresses within the cross-section of the steel member, Jönsson and Stan [13]. As the model only accounts for different cross-section types and susceptible axis of buckling, it is not capable to change the magnitude or distribution of the residual stress. Where the model created at Lehigh University is based upon stress magnitudes within the cross-section, the Ayrton-Perry model accounts for residual stresses by equating them as imperfections. The article presents and reviews the fundamentals of the Lehigh and Ayrton-Perry models. It further modifies these in order for them

[4] Madsen, I: *Box Girder Buckling Tests*, 1941, Tech. Report 193.14, Lehigh University, Pennsylvania, USA.

to account for both imperfections and residual stresses by creating new models modified according to the original. All models are thoroughly discussed and compared. As a final consideration, efforts have been made in investigating the possibilities of establishing a versatile model by regarding the dissipated energy from the partially yielded cross-section.

2. Review of residual stress

2.1 Fundamentals

In the field of solid mechanics, the notion of mechanical stress can be perceived as the resistance exerted by a body's inter-atomic bonds when its molecular structure is distorted. Thus, if a temporary process dislocates a body's atomic structure in a manner such that permanent incompatible strain fields are formed, internal stress fields will form due to the permanent straining of the atomic bonds. Simply stated, if a body is subjected to strains, an internal system of stress fields are necessary in order to satisfy equilibrium and compatibility. These stress fields are in the present article formally defined as residual stresses. As the present study regards structural grade steel members, notable processes from which residual stresses arise are

- Welding (Built-up cross-sections)
- Plastic Deformation (Cold bending)
- Differential cooling (Hot-rolled cross-sections)

Further, the models used in this study base the numerical values of the residual stress distribution from an experimental study of an as-delivered hot-rolled H300B S275JR structural steel grade specimen. As such, the primary process is that of differential cooling as no cold ending lines were observed. The study is fully covered in Andersen and Kabel [13, 14], where the residual stress distribution were obtained by the method of sectioning. The obtained results from the experiment is presented in figure 2.2. Here, a notable property of the residual stress distribution is that the flange tips contain compressive residual stresses. This behaviour corresponds well with experimental data from Lehigh University, see e.g. Beedle and Huber [2], Fujita [9], Huber and Beedle [12]. In addition, it corresponds well with an analysis of the differential cooling rate which a wide-flange steel specimen undergoes after being rolled. As shown in Huber [11], the area of the cross-section prone to cool at the slowest rate will be in a state of tension and vice versa. A more rigorous and comprehensive treatment on the fundamentals of residual stress can be found in Andersen and Kabel [16].

2.2 Properties of residual stress

An important property of residual stress is the fact that it is present in a body absent external loads and thermal gradients. As such, if we consider a cross-section with some arbitrary residual stress distribution $\sigma_r(y,z) = \sigma_{ry}(y) + \sigma_{rz}(z)$ - see figure 2.1 – the axial force equilibrium read

$$\int_A \sigma_r(y,z)\mathrm{d}A = 0$$
$$2t_f\int_{-b/2}^{b/2}\sigma_{ry}(y)\mathrm{d}y + t_w\int_{-h/2}^{h/2}\sigma_{rz}(z)\mathrm{dz} = 0 \tag{2}$$

and force moment equilibriums

$$\int_A \sigma_r(y,z)y\mathrm{d}A = 0$$
$$2t_f\int_{-b/2}^{b/2}\sigma_{ry}(y)y\mathrm{d}y = 0 \tag{3}$$

$$\int_A \sigma_r(y,z)\mathrm{z}\mathrm{d}A = 0$$
$$t_w\int_{-h/2}^{h/2}\sigma_{rz}(\mathrm{z})\mathrm{zdz} = 0 \tag{4}$$

Consequently, as residual stress is required to be self-equilibrating no immediate deformation of the body will occur, rendering the process of determining the magnitude and distribution of these troublesome. Further, as the distribution and magnitude of residual stress are highly sensitive to many factors, the process of accurately estimate them is difficult.

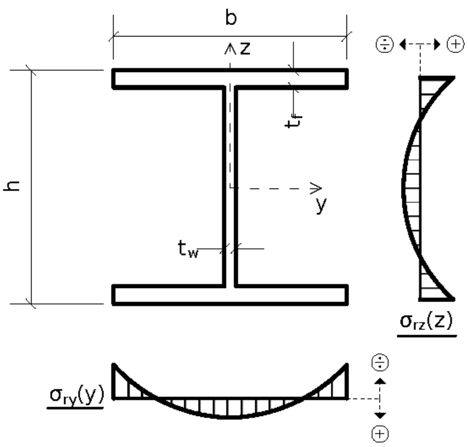

Figure 2.1 Equilibrium condition for longitudinal residual stress

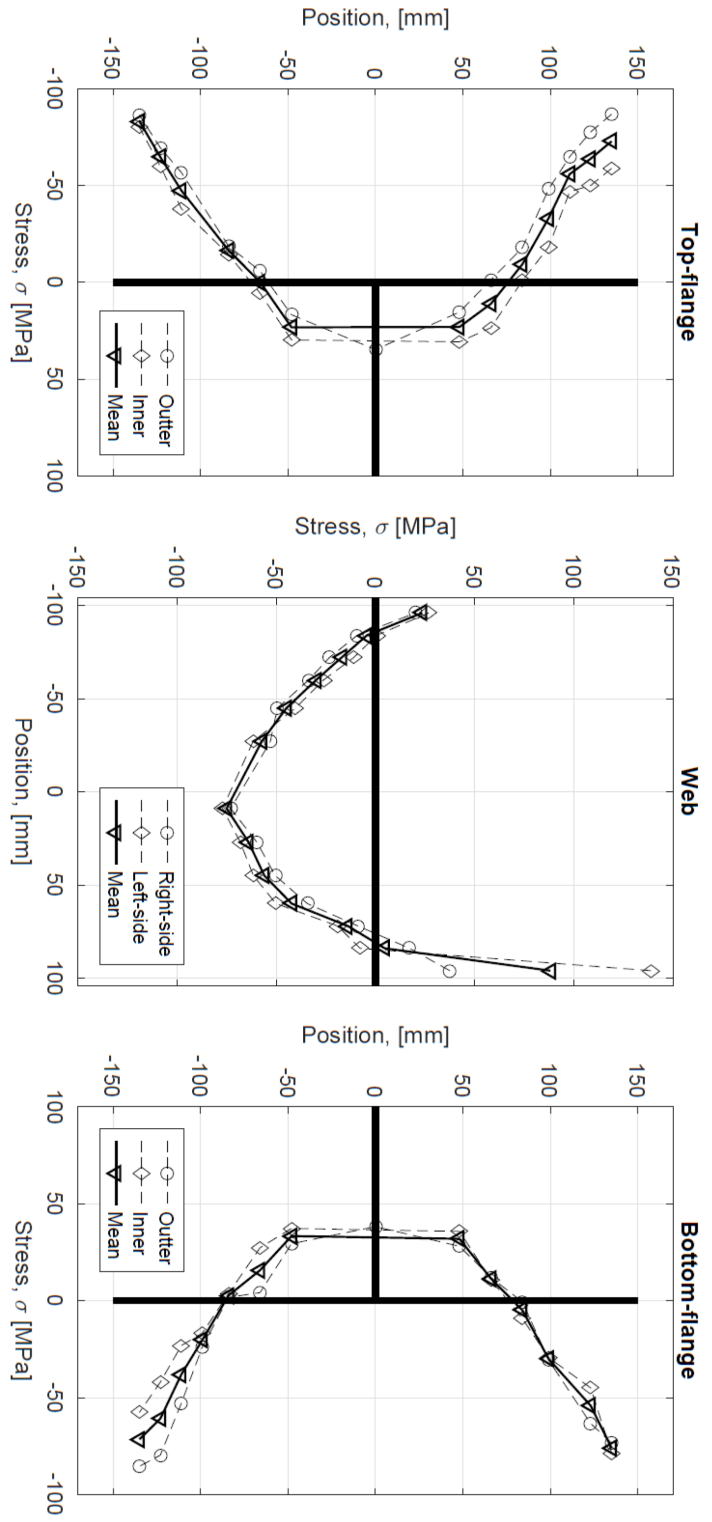

Figure 2.2 Residual stress distribution of an HE300B wide-flange steel specimen, see Andersen and Kabel [13, 14]

Another important property is that residual strains superimpose exactly as strains arising from external influence in the elastic range as noted by Lee [17]. Thus, it is evident that residual stress has a definite influence on the stress-strain response behaviour of a steel member. Consequently, the linear-elastic stress-strain behaviour of a stub column, containing residual stress will cease to be valid prior to reaching the yield limit of the entire cross-section. Assuming no imperfections are present, and the column has a yield stress level of σ_Y with a maximum compressive stress at its flange tips of σ_{rc}, a proportionality stress level, σ_p, can be expressed by

$$\sigma_p = \sigma_Y - \sigma_{rc} \tag{5}$$

The proportionality limit thus coincide with the onset at which the columns stress-strain behaviour becomes non-linear. This behaviour is of great importance when regarding stub columns, as their stability resilience is greatly compromised.

Based on these regards, it is possible to obtain the relative slenderness ratio at which the non-linear buckling will occur, here denoted λ_p. For columns with a slenderness ratio < λ_p caution should be taken in the design. Assuming the column contains no imperfections, and the column support condition is pinned-pinned, Euler's formula is

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \tag{6}$$

Thus, the proportional slenderness ratio is obtained when $\sigma_{cr} = \sigma_p = \sigma_Y - \sigma_{rc}$ as

$$\lambda_p = \sqrt{\frac{\pi^2 E}{\sigma_Y - \sigma_{rc}}} \tag{7}$$

As stated by Jönsson and Stan [13], it is customary to apply a maximum compressive residual stress value at the flange tip of $0.3\sigma_Y$ if the ratio of the largest cross-sectional dimension is $h/b > 1.2$ and $0.5\sigma_Y$ if the ratio is < 1.2. This is supported by ECCS [5, 7, 8] which have used experimental work from Schulz[5]. However, experiments have shown that the maximum compressive residual stress value is independent of σ_Y, but rather attain a value of approximately $0.5\sigma_Y^*$, where σ_Y^* = 235 MPa, [1, 16, 17]. Thus, a general expression for the onset at which non-linear buckling would occur, can be expressed by

[5] Schulz, G. *The maximum strength of axially loaded columns considering geometrical imperfections and material inhomogeneities.* Thesis, Technical University Graz, Graz, 1968

$$\lambda_p = \sqrt{\frac{\pi^2 E}{\sigma_Y - 0.5\sigma_Y^*}} \tag{8}$$

Using the experimentally obtained values from Andersen and Kabel [13, 14] for σ_Y and E as presented in table .1, the value of the proportional slenderness ratio, according to eq. (8), becomes 96. Alternatively, using the experimentally determined maximum compressive residual stress, σ_{rc}, also presented in table 3.1, and applying eq. (7) the relative slenderness ratio becomes 90. This indicate the use of $0.5\sigma_Y$* as an indicator of the maximum compressive residual stress is a little conservative.

3. Lehigh Model

In 1954, as a part of a extensive research project conducted at Lehigh University, Huber and Beedle published a model in [12] for a columns flexural buckling strength in which the influence of residual stresses were incorporated. The model is henceforth referred to as the *Lehigh model*. As will be evident, this model is based on an analytical approach, as opposed to the semi-empirical models proposed by Engesser.

Basically, the model seeks to relate the average critical stress with the yield stress level as a function of the plastically strained area of the cross-section. This is achieved by regarding the requirement of axial force equilibrium under the following assumptions

- The column does not fail prior to the outermost fires of the flanges have yielded, $\sigma_{rc} \geq \sigma_p$
- The elastic area of the cross-section always remains double symmetrical
- The residual stress distribution of the top and bottom flange is equal
- The linear-elastic range is unaffected by residual stresses
- The column has no initial geometrical imperfections

Preliminarily, the model requires knowledge of the materials yield level, $\sigma_{Y,}$ and the extremums of the residual stress distribution.

3.1 The basic model theory

The following is based on the nomenclature presented in figure 3.1. As observed, the only variable is y_0 as $z_0(y_0)$, which controls the extent of the yielded area of the cross-section. Thus, a numerical value of the critical stress for each incrementation of the yielded area of the cross-section is obtained. By modifying Euler's formula to account for the reduced flexural stiffness of the cross-section, it is possible to obtain corresponding slenderness ratios.

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \frac{I_e}{I} \qquad \lambda = \frac{l_s}{i} \tag{9}$$

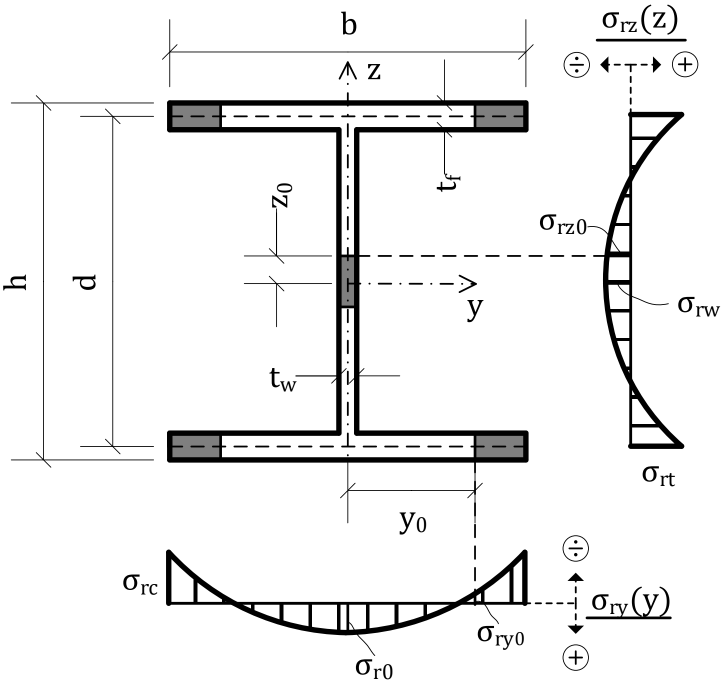

Figure 3.1 Nomenclature for establishing the Lehigh model

The general principal of the approach is to require axial force equilibrium where the internal forces are divided into four contributors; elastic and plastic forces of the flanges, and elastic and plastic forces of the web. Thus, if the axially applied load is denoted P_0, axial equilibrium reads

$$P_0 = \Delta P_{f,e} + \Delta P_{w,e} + \Delta P_{f,p} + \Delta P_{w,p} \tag{10}$$

Where the index indicates either flange, *f*, or web, *w*, and elastic, *e*, or plastic, *p*. Depending on the extremums of the residual stress distributions, either the flanges yield prior to the web initiate yielding or the web initiate yielding prior to the flanges have fully yielded. As the following examples will be based on residual stress extremums determined in Andersen and Kabel [13, 14] and as presented in figure 2.2, the web will initiate yielding prior to the flanges have yielded, $\sigma_{rw} < \sigma_{ro}$. As such, only this case is covered in the present article. A more thorough treatment of the approach, including the latter case, is given in Andersen and Kabel [16]. The different stress-states and the gradually developing yielded regions are diagrammatically illustrated in figure 3.2.

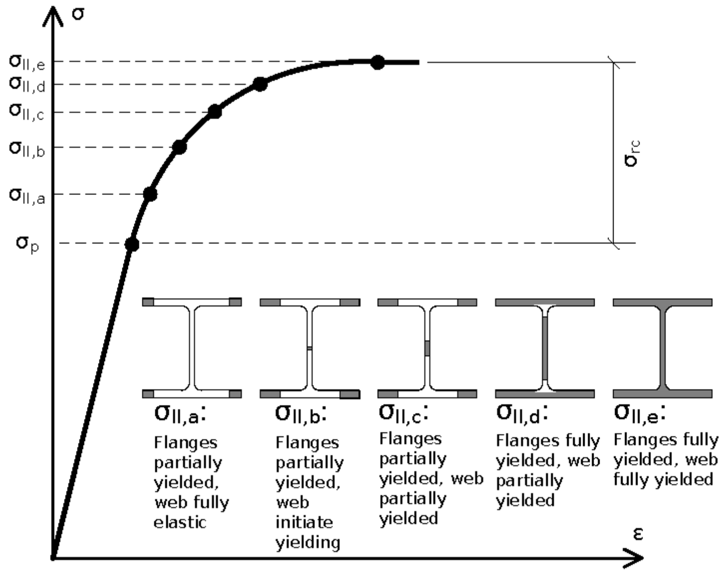

Figure 3.2 Stress-states and gradually development of yielded regions

As seen, the only necessary governing equations are for the stress states of $\sigma_{II,a}$ and $\sigma_{II,c}$. The others are simply limits of these. Applying eq. (10), these stress states are governed by the following equation.

$$\sigma_{II,a} = \sigma_Y - \frac{A_e}{A}\sigma_{ry_0} - \frac{4t_f}{A}\int_{y_0}^{b/2}\sigma_{ry}\mathrm{dy} \tag{11}$$

$$\sigma_{II,c} = \sigma_Y - \frac{A_e}{A}\sigma_{ry_0} - \int_{y_0}^{b/2}\sigma_{ry}\mathrm{dy} - \int_0^{z_0}\sigma_{rz}\mathrm{dz} \tag{12}$$

3.2 Residual stress distributions

As is evident from equations (11) and (12) an expression for the residual stress distribution is required. The immediate versatility of the *Lehigh model* is that if the distribution can be formulated as a function of *y* or *z*, it can be employed in the model. In the present study eight of these expressions have been established and are

Linear Distribution

$$\sigma_{ry} = \frac{2}{b}\left(\sigma_{rc} - \sigma_{ro}\right)y + \sigma_{ro}$$

Parabolic Distribution

$$\sigma_{ry} = \frac{4}{b^2}\left(\sigma_{rc} - \sigma_{ro}\right) y^2 + \sigma_{ro}$$

$$\sigma_{ry} = \frac{2}{b^2}\left(\sigma_{rc} - \sigma_{ro}\right) y^2 + \frac{1}{b}\left(\sigma_{rc} - \sigma_{ro}\right) y + \sigma_{ro}$$

$$\sigma_{ry} = \frac{4}{b^2}\left(\sigma_{rc} - \sigma_{ro}\right)\left(y - \frac{b}{2}\right)^2 + \sigma_{rc}$$

Cubic Distribution

$$\sigma_{ry} = \frac{8}{b^3}\left(\sigma_{rc} - \sigma_{ro}\right) y^3 + \sigma_{ro}$$

$$\sigma_{ry} = -\frac{2}{3}\frac{\left(2\sigma_{rc} - 3\sigma_{ro}\right)}{b^3} y^3 - \frac{\sigma_{ro}}{b^2} y^2 + \frac{1}{3}\frac{\left(7\sigma_{rc} - 6\sigma ro\right)}{b} y + \sigma_{ro}$$

Cosine Distribution

$$\sigma_{ry} = \sigma_{ro}\cos\left(\frac{2\pi}{b} y\right) + \frac{2}{b}\left(\sigma_{rc} - \sigma_{ro}\right) y$$

Square Wave Distribution

$$\sigma_{ry} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\frac{\sigma_{ro} - \sigma_{rc}}{2}}{2n-1}\sin\left(\left(2n-1\right)\frac{4\pi}{b} y\right) + \frac{\sigma_{ro} + \sigma_{rc}}{2}$$

As the model always assume a double symmetrically elastic cross-section, it is required that the residual stress distributions are symmetrical around the axis perpendicular to their variational axis. Thus, σ_{ry} is defined for every $y \in [0, b/2]$ and σ_{rz} for every $z \in [0, d/2]$. Furthermore, they are easily modified to be employed as residual stress distribution functions for the web, σ_{rz}, by replacing the constant b, σ_{rc}, σ_{ro} with d, σ_{rw}, σ_{rt} and the variable y with z.

3.3 Numerical exemplification

Based on the numerical values given in table 3.1, examples are presented. The geometric properties are the nominal values of an HE300B type cross-section, and the extremums are those obtained experimentally in Andersen and Kabel [13, 14] including a mean yield stress level and elastic modulus. Using the expression given in section 3.2, a conjoined plot of all eight assumed residual stress distributions are plotted for $y \in [0, b/2]$ in figure 3.3.

Table 3.1 Numerical values for exemplification

σ_{rc}	σ_{ro}	σ_{rt}	σ_{rw}	σ_{Y}	E
		[MPa]			
-87	35	35	-73	-329	199×10^3
h	b	d	t_f	t_w	A
		[mm]			[mm^2]
300	300	281	19	11	14491

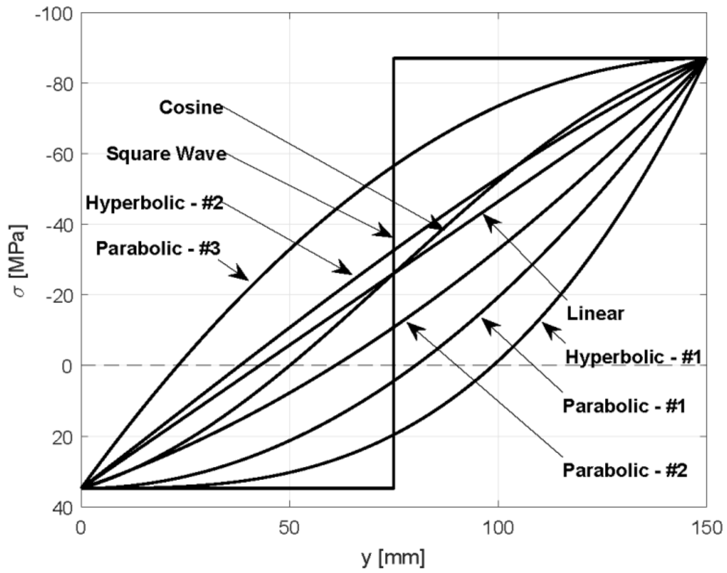

Figure 3.3 Assumed residual stress distributions from section 3.2

Implementing the residual stress distributions in eqs. (11) and (12) critical stress levels are obtained for each increment of y_0. By these it is possible to determine the corresponding slenderness ratio (λ) for each critical stress by using a *modified Euler column equation*, which is the Euler formula multiplied by the ratio between the elastic and the total area moment of inertia, i.e. I_e and I respectively.

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2}\frac{I_e}{I} \Rightarrow \lambda = \sqrt{\frac{\pi^2 E}{\sigma_{cr}}\frac{I_e}{I}} \quad (13)$$

Figures 3.4 and 3.5 are plots of the influence of the individual residual stress distributions on an HE300B cross-sections buckling strength around its major axis.

Figures 3.6 and 3.7 are plots of the influence of the individual residual stress distributions on an HE300B cross-sections buckling strength around its minor axis.

As can initially be observed in figures 3.4 to 3.7, the influence of residual stress can severely reduce the buckling strength of a column. Further, a logical behavior is observed by regarding the individually assumed residual stress distributions influence. The distributions imposing the highest compressive residual stresses are lower than those with a smaller area. This is explained in the rate at which the flanges yield for every increment of y_0. All of the different diagrams intersect with equation (13) in the same point which is at the value of equation (5), which is the proportionality limit as described earlier in the section. Below this value, the column is effected of such small value of stress, that the column will not undergo plastic deformation and instead follow the elastic formula put forth by Euler.

The analysis is carried out until the flanges yielded completely. In general a large area of compressive residual stresses in the flanges results in a lower column curve, same as seen in Huber and Beedle [12].

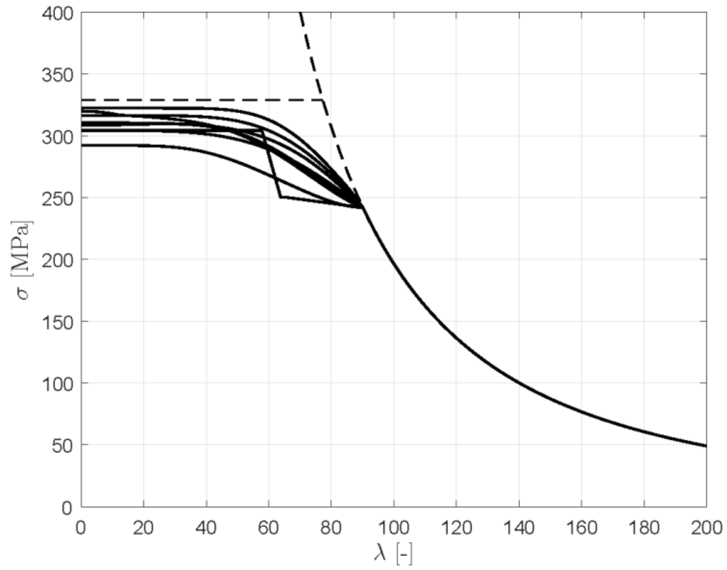

Figure 3.4 The influence of the assumed residual stress distributions from section 3.2 on the buckling strength of an HE300B - major axis

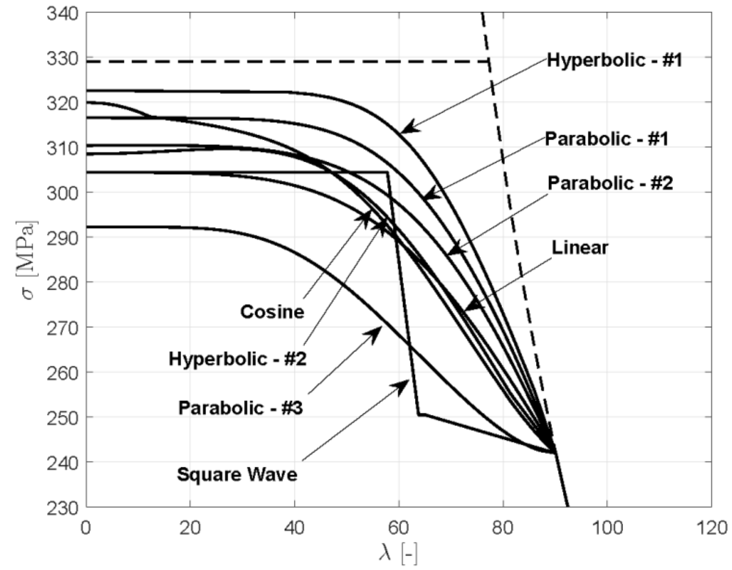

Figure 3.5 A close-up of the influence of the assumed residual stress distributions from section 3.2 on the buckling strength of an HE300B - major axis

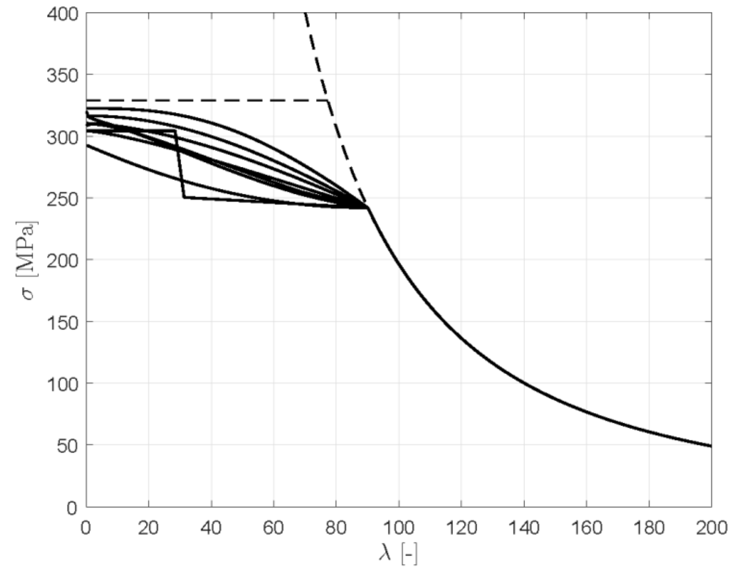

Figure 3.6 The influence of the assumed residual stress distributions from section 3.2 on the buckling strength of an HE300B - minor axis

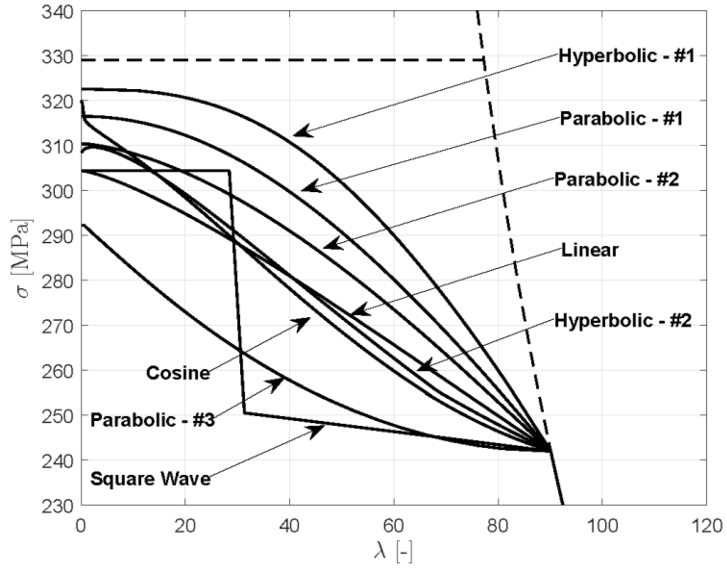

Figure 3.7 A close-up of the influence of the assumed residual stress distributions from section 3.2 on the buckling strength of an HE300B - minor axis

4. Ayrton-Perry model

In 1886 Ayrton and Perry analysed axial and concentrically loaded columns with initial curvature. This approach has later been named the *Ayrton-Perry method*, and is today the background method for the establishment of a columns buckling strength diagrams in the European Standard. Their assumptions were very simple, and consisted only of a pin-ended elastic column with no residual stress, but with an initial deflection. The residual stress is later introduced in ECCS [7] as by equating it as an geometrical imperfection. The expression is obtained by considering a column under combined bending and axial load, where the bending is applied by an initial deflection, u_i, see figure 4.1. The initial deflection is assumed to be formed based on a sinus-wave, with the amplitude given as a. Using the definitions of figure 4.1, the initial deflection is governed by the equation.

$$u_i\left(x\right)=a\sin\left(\frac{\pi}{L}x\right) \tag{14}$$

Using that there is no horizontal force, but the bending is given due to the initial deflection, the limit for the column is given as

$$\frac{N}{A}+\frac{Nu}{W}=\sigma_Y \tag{15}$$

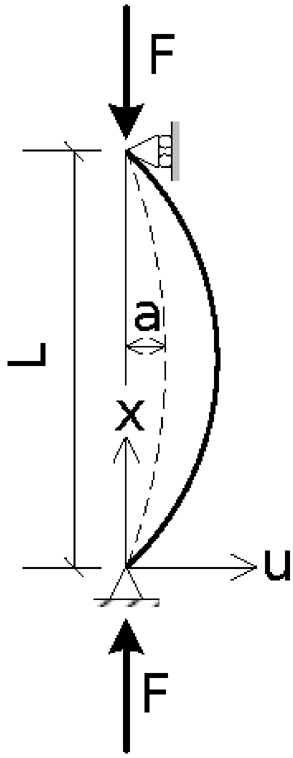

Figure 4.1 Column curve under bending and axial load

Where *A* and *W* is the cross-sectional area and the section modulus respectively. By introducing both σ_b as the normal stress and σ_{cr} as the *Euler stress*, together with obtaining the particular solution for the *Euler-Bernoulli beam theory* for the deflection, *u*, it is seen that it can be rewritten

$$(\sigma_{cr}-\sigma_b)(\sigma_Y-\sigma_b)=\sigma_b\sigma_{cr}u_i\frac{A}{W}$$
$$(\sigma_{cr}-\sigma_b)(\sigma_Y-\sigma_b)=\sigma_b\sigma_{cr}\eta,\quad \text{where}\quad \eta=u_i\frac{A}{W} \tag{16}$$

where the η-factor represents the straightness of the column, thereby being an imperfection factor that can include geometrical imperfections such as initial deflection, but also residual stresses or tolerance of rolling. By introducing a column reduction factor, χ, along with a relative slenderness, λ*, as

$$\chi=\frac{\sigma_b}{\sigma_Y}\quad\wedge\quad\lambda^*=\sqrt{\frac{\sigma_Y}{\sigma_{cr}}}\quad\text{where}\quad\sigma_{cr}=\frac{\pi^2E}{\left(\frac{l_s}{i}\right)^2} \tag{17}$$

This then allows for the following equation

$$\lambda^{*2}\chi^2 - \chi(\lambda^{*2} + \eta + 1) + 1 = 0 \tag{18}$$

According to DS/EN-1993 [4], the coefficient is given as $\eta = \alpha\ (\lambda^* - \lambda_0^*)$, where λ_0^* = 0.2 due to the buckling effects may be ignored beneath these values, and to allow for the effect of strain-hardening, see ECCS [5]. By introducing this into equation (18), gives

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \lambda^{*2}}} \quad \text{where } \phi = \frac{1}{2}\left(\lambda^{*2} + \alpha(\lambda^* - 0.2) + 1\right) \tag{19}$$

Thus the reduction factor, χ, can be calculated. Here the coefficient of α is dependent on which buckling curve is used. It takes both geometric and mechanical imperfections into account. The value of α range from 0.13 – 0.76 in DS/EN-1993 [4]. This range is based on the initial deflection of L/ξ for ξ ranging from 150 to 350. If only the geometric imperfections were to be used, then the column formulas would be based on an initial deflection, $u_1 = L/\xi = L/1000$, according to Bonnerup *et al.* [3].

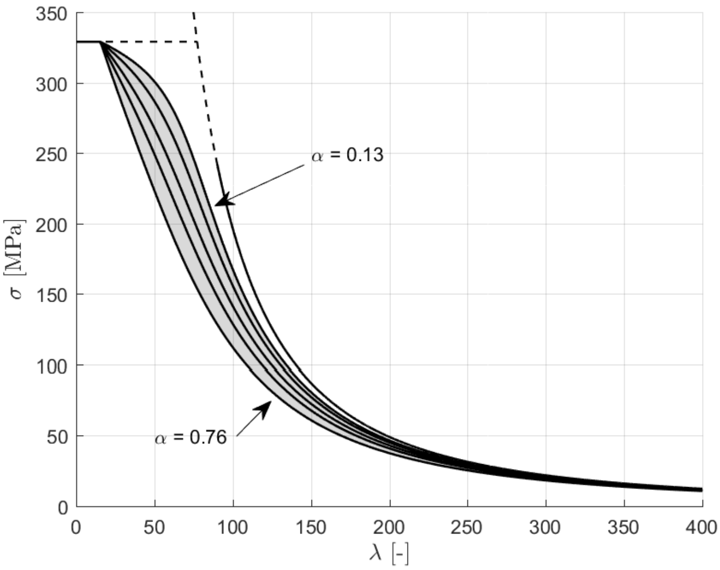

Figure 4.2 Diagram of relation between column curve according to Eurocode and the obtained residual stress

Applying equation (19) for a given λ, a reduction factor is obtained, thus a critical stress value for the Ayrton-Perry model is obtained as $\sigma_{cr,A} = \chi\sigma_Y$. Hereby, a comparison between the Lehigh model and the Ayrton-Perry model is possible. The different buckling strength curves, for α ranging from 0.13 to 0.76 is given in figure 4.2. Here the grey-shaded area constitutes the range of the buckling strength curves, in accordance with DS/EN-1993 [4].

5. Modified Lehigh

To compare the Ayrton-Perry model with the Lehigh model, a common background is necessary. As the Lehigh model assumes that the column system contains no initial geometrical imperfection, they are not directly comparable. Thus, it is modified to account for these in the following.

The modification of the model simply superimposes the residual stresses and the stresses arising from this assumed initial deflection. The modification assumes that stresses arising from the initial deflection can be equated as bending stresses. Two general cases of bending stress can arise from imperfection. Namely in bending stress from imperfections around the major or minor axis. These general cases are sketched in figures 5.1 and 5.2. Here, an initial bow imperfection introduces a bending moment, as the force, *P*, acts at an eccentricity, *e*, from the center of the web.

The preliminary assumptions of the model follow those stated in the Lehigh model, except the assumptions of the column being perfectly straight and the double symmetric elastic area. Further, it is assumed that the bending stress will be carried in the flanges only. Thus, the flanges will have a superimposed bending stress that is either constant or linear distributed.

For Case 1, the constant stress is equated over the flanges. As such, the applied stress will be shifted by a constant stress magnitude as either compression or tension. Thus, the maximum compressed flange and the web will initiate yielding prior to the other flange, thus complicating the model significantly. A sketch of the concept is given in figure 5.1. Ultimately, the flange with least compressive stress yields as the last part. Therefore, when calculating the stress summation, the calculations need to be divided into the different sections that becomes plastic as a stress is applied. For Case 2, the stress is linearly distributed over the flanges with a compression and tension part. The result of which creates a scenario where the flanges yields in one of the flange tips while the other remain elastic. An illustration is seen in figure 5.2. Thus, a skewer end-result is given, where a flange yields at an increased rate in one half of its length, however with the same rate as the other flange. Therefore, the calculations are to reflect such differences in sections where some part of the flange becomes plastic, while other parts stay elastic. Both cases are based on the approach presented in section 3.1 and equation (10), i.e.

$$P_0 = \underbrace{\overbrace{\Delta P_{f,p} + \Delta P_{w,p}}^{\text{Plastic part}} + \overbrace{\Delta P_{f,e} + \Delta P_{w,e}}^{\text{Elastic part}}}_{\text{Total force from flange and web}} \tag{20}$$

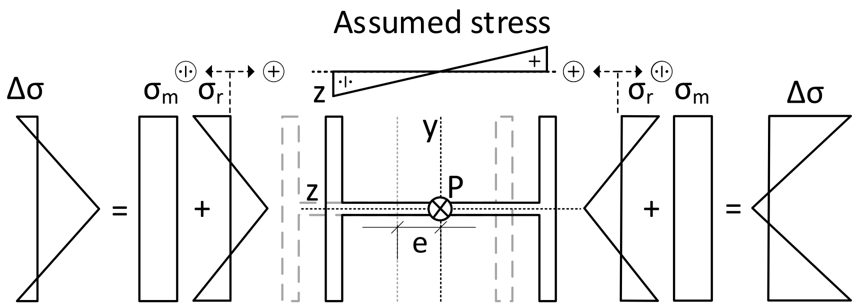

Figure 5.1 Case 1 - Bending around the major axis

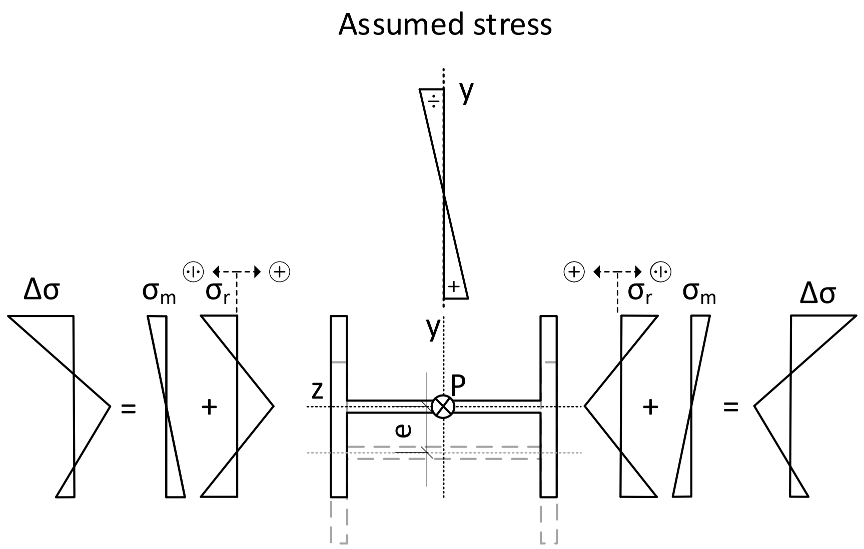

Figure 5.2 Case 2 - Bending around the minor axis

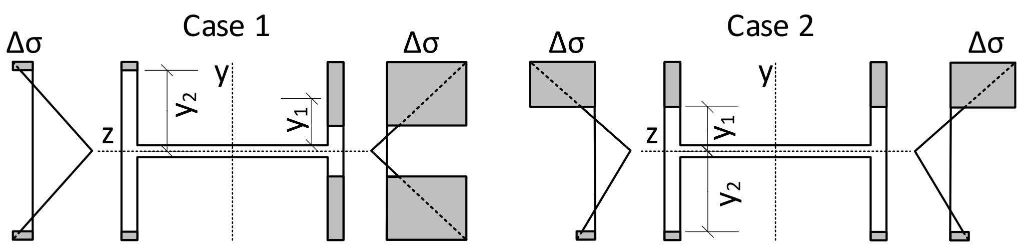

Figure 5.3 Basic nomenclature used in the derivation of the modified Lehigh-model

The method presented in section 3.1 was based on a single variable, y_0, as the elastic area of the cross-sections were symmetrical. Due to the skewness of the applied stress, two new variables are instead introduced, y_1 and y_2, where y_1 indicates the first area to become plastic. It is illustrated in figure 5.3. Consequently, the axial force equilibrium ultimate is expressed as

$$\begin{aligned}\sigma = \frac{2t_f}{A}\int_{y_1}^{b/2}\left(\sigma_Y - \sigma_{ry} - \sigma_M\right)\mathrm{d}y + \frac{2t_f}{A}\int_{y_2}^{b/2}\left(\sigma_Y - \sigma_{ry} + \sigma_M\right)\mathrm{d}y \\ + \frac{2t_f y_1}{A}\left(\sigma_Y - \sigma_{ry_1} - \sigma_M\right) + \frac{2t_f y_2}{A}\left(\sigma_Y - \sigma_{ry_2} + \sigma_M\right) + \frac{t_w d}{A}\left(\sigma_Y - \sigma_{ry_1}\right)\end{aligned} \tag{21}$$

where σ_M indicates the stress from the moment caused by the eccentric load, which is defined later. Carrying out the integrals thus yield a familiarity with the expressions derived in the Lehigh model.

$$\begin{aligned}\sigma = \sigma_Y - \frac{2t_f}{A}\int_{y_1}^{b/2}\left(\sigma_{ry} + \sigma_M\right)\mathrm{d}y - \frac{2t_f}{A}\int_{y_2}^{b/2}\left(\sigma_{ry} - \sigma_M\right)\mathrm{d}y \\ - \frac{2t_f y_1}{A}\left(\sigma_{ry_1} + \sigma_M\right) - \frac{2t_f y_2}{A}\left(\sigma_{ry_2} - \sigma_M\right) - \frac{t_w d}{A}\left(\sigma_{ry_1}\right)\end{aligned} \tag{22}$$

This expression is given for the situation where the most compressed flange has fully yielded prior to the web initiate yielding i.e. $\sigma_{ro} < \sigma_{rw}$. However, for the case where $\sigma_{rw} < \sigma_{ro}$ the web initiate yielding prior to the most compressed flange has fully yielded. For this situation, the following function is used

$$\begin{aligned}\sigma = \sigma_Y - \frac{2t_f}{A}\int_{y_1}^{b/2}\left(\sigma_{rx} + \sigma_M\right)\mathrm{d}y - \frac{2t_f}{A}\int_{y_2}^{b/2}\left(\sigma_{ry} - \sigma_M\right)\mathrm{d}y \\ - \frac{2t_f y_1}{A}\left(\sigma_{ry_1} + \sigma_M\right) - \frac{2t_f y_2}{A}\left(\sigma_{ry_2} - \sigma_M\right) \\ - \frac{2t_w}{A}\int_0^{z_0}\sigma_{rz}\mathrm{d}z - \frac{t_w(d-2z_0)}{A}\left(\sigma_{rz_0}\right)\end{aligned} \tag{23}$$

where another assumption is given as $\sigma_{rz0} = \sigma_{ry1}$. For the situation it is also given that $y1 \wedge y2 \leq d/2$. The σ_M-part of the equations is dependent of which of the earlier mentioned cases that are given. These would be

$$\text{Case 1:} \quad \sigma_M = \sigma_m \tag{24}$$

$$\text{Case 2:} \quad \sigma_M = \sigma_m y \tag{25}$$

where σ_m is the extremum value of the stress arising from the initial deflection. This value is found by calculating a force, P, for initial deflection of L/ξ for ξ =

1000, as DS/EN-1993 [4] recommend this value for a column without residual stresses. Thus give

$$\sigma_m = \frac{Pe}{W} = \frac{\pi h}{2\xi}\sqrt{\frac{EA}{I}}\sqrt{\sigma} \tag{26}$$

where it is used that $P = \sigma A$, $L = \lambda i$ and $\lambda = \pi(E/\sigma)^{0.5}$. Using this equation, it is possible to rewrite equation (22) and (23). However, this also means that the equations are dependent upon the result itself, why an iteration process is needed in order to obtain a result. In this study, this is only carried out for Case 1 with the residual stress distribution assumed linearly distributed.

In order for the calculation to be possible, it is important to obtain multiple equations, due to the variables being σ, σ_m, I, y_1, y_2. All of these variables will vary for each new increment of value z_0, meaning more and more of the section becomes plastic. So, for the 5 variables, 5 equations needs to be obtained. This is fulfilled by first looking at the assumption that the value of stress in each of the flanges needs to be the same.

$$\sigma_{ry_1} = \sigma_{ry_2} = \sigma_{rz_0} \tag{27}$$

which gives the following equations

$$y_1 = \frac{b}{d}\frac{(\sigma_{rt} - \sigma_{rw})}{(\sigma_{rc} - \sigma_{ro})} z_0 - \frac{b}{2}\frac{(\sigma_{rw} - \sigma_{ro} - \sigma_m)}{(\sigma_{rc} - \sigma_{ro})} \tag{28}$$

$$y_2 = y_1 - b\frac{\sigma_m}{(\sigma_{rc} - \sigma_{ro})} \tag{29}$$

These, together with the already given equations (23) and (26) and the geometrical boundary of either

$$I_{ey} = \frac{t_f}{6}\left(3d^2 + t_f^2\right)(y_1 + y_2) + \frac{t_w}{12}\left(d^3 - 8z_0^3\right) \tag{30}$$

$$I_{ez} = \frac{2t_f}{3}\left(y_1^3 + y_2^3\right) + \frac{t_w^3}{12}\left(d - 2z_0\right) \tag{31}$$

results in 5 equations with 5 variables. Thus, the calculation of a critical stress value is possible to obtain by the modified Lehigh model. From this a new slenderness ration can be calculated. This, along with the same linear-relation found previously is given underneath in figure 5.4.

Here it is seen that the addition of a force moment, that is related to the stress and an initial deflection of *L/1000* reduces the values of the stress, thereby reducing the capacity of the column. It is seen that the overall value of the modified model is below the original model, which shows that an initial deflection will reduce the capacity value. If small calculations are done for the extremum of either full plastic (λ_0, where $\lambda = 0$) or full elastic (λ_3, see point 3 on figure 5.4), it is seen that

$$\lambda_0: \quad \sigma_0 = \sigma_Y - \frac{2t_f b}{A}\left(\frac{1}{2}(\sigma_{rc} + \sigma_{ro})\right) - \frac{t_w d}{A}\left(\frac{1}{2}(\sigma_{rt} + \sigma_{rw})\right) \tag{32}$$

$$\lambda_3: \quad \sigma_3 = \sigma_Y - (\sigma_{rc} - \sigma_m) \tag{33}$$

where it is given that the stress-state at a fully yielded cross-section is the yield stress minus a mean value of the given residual stresses within the profile, as was the case for the original Lehigh model. The case for a fully elastic cross-section it is given by $\sigma_Y - \sigma_{rc}$ - σ_m, consequently decreasing the stress level of the proportionality limit by a value depending on the initial deflection. For point 1 in figure 5.5, it is given that the cross-section is fully elastic in the flange, where the initial deflection causes a rise in tensile stress. At point 2, the web also becomes fully plastic, so that it is only the remaining flange that yields, which changes in point 3, where it becomes fully elastic, see figure 5.5. Point 0 is at $\lambda = 0$, where the cross-section is completely plastic.

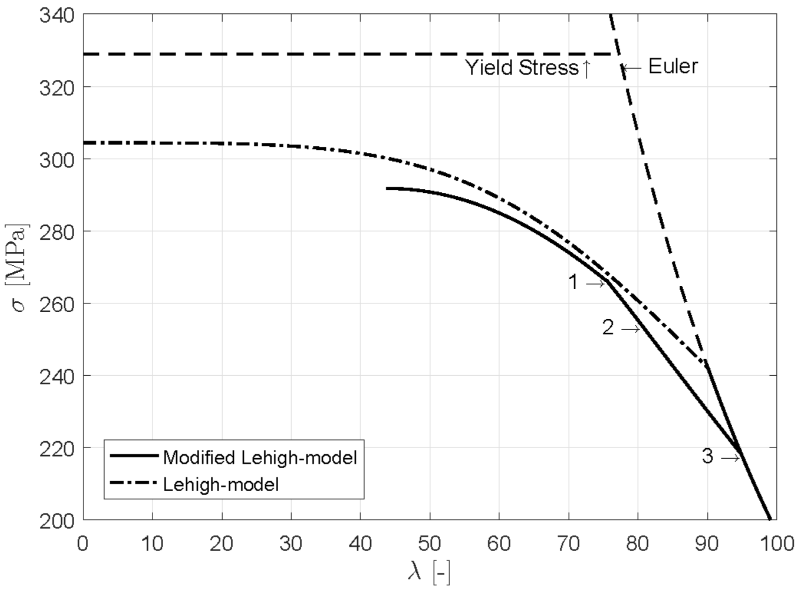

Figure 5.4 Buckling strength of an HE300B cross-section - major axis

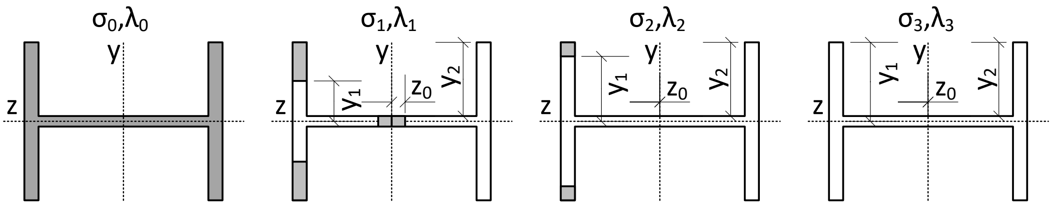

Figure 5.5 The three stages as marked in figure 5.4

6. Modified Ayrton-Perry

Where section 5 modified Lehigh's model to account for geometrical imperfections, this section regards the results when the Lehigh-model's influence of residual stress is combined with the Ayrton-Perry model's influence of an initial deflection. It is evident from figure 4.2, that an imperfect column will have a reduced buckling strength, even for linear buckling. Thus, the onset for the influence of residual stresses should coincide with the given value for an imperfect column for $\xi = 1000$. Thus a combination of the models is carried out, which for the sake of convenience is referred to as the *modified Ayrton-Perry model*. Thus, at the range of linear buckling only geometrical imperfections influence the buckling strength, whereas the non-linear buckling strength is influence by both geometrical imperfection and residual stresses.

The approach considers the reduction in buckling strength obtained of both the Ayrton-Perry model and the Lehigh model in relation to either. Thus

$$\sigma_{\text{mod}}(x) = \begin{cases} \sigma_{\text{imp}} - (\sigma_Y - \sigma_{rs}) & x \in [0 : \lambda_E] \\ \sigma_{\text{imp}} - (\sigma_E - \sigma_{rs}) & x \in [\lambda_E : \lambda_{rs}] \\ \sigma_{\text{imp}} & x \in [\lambda_{rs} : \infty] \end{cases}$$

where the three cases are plotted in figure 6.1 by the numbers 1, 2, 3. The symbol σ_E is defined as the Euler stress, and described on the next page.

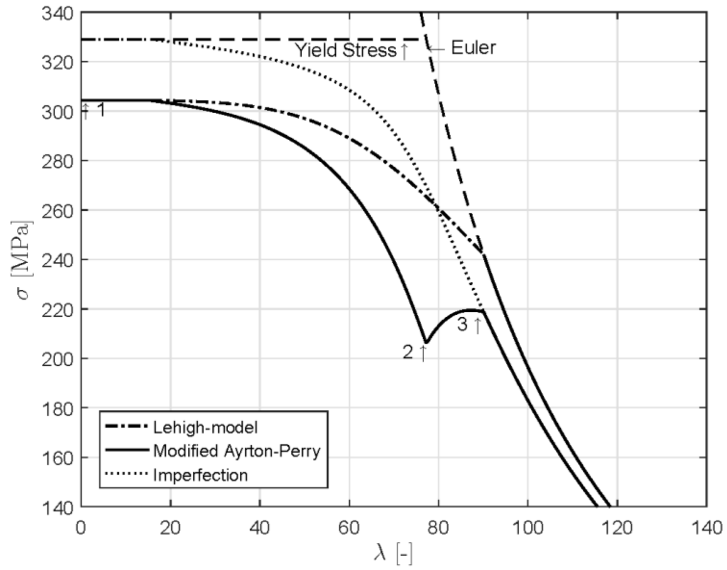

Figure 6.1 Comparison of the Lehigh, imperfection and the modified Ayrton-Perry model

It is given that σ_{rs} is the given value for a linear distribution of residual stress in accordance with the Lehigh-model. σ_{imp} is given as the stress in accordance to the geometrical imperfection given by *L/1000*. This function is obtained using already known values of α in accordance with DS/EN-1993 [4], in which there is given different α-values for each column curve. It is used that for *a,b*-curve, the value is $\alpha = 0.21 \wedge 0.24$, and they corresponds with a value of *L/300 – L/250* for elastic behaviour and *L/250 – L/200* for plastic behaviour. Thus, a linear extrapolation for the value of α for *L/1000* is; *1000/250 = 4* thus $\alpha = 0.21/4 \approx 0.05$. This enables a calculation of σ_{imp}, which is the stress due to pure geometrical imperfection. Further, it is given that the value of the stress for an Euler column is denoted σ_E. The value of stress is dependent on the value of slenderness ratio, λ. Due to the two models always being a reduction of stress in accordance with the maximum stress, the function changes over the diagram. As seen, it is given by a reduction in relation to the yield stress, σ_Y, up to the slenderness ratio of λ_E, which is where the yield stress no longer is constant, and instead follows the column curve from Euler. Hereafter, the reduction is always in relation to the given value of an Euler-column. The function follows this relation until it reaches λ_{rs}, where the residual stress no longer has an influence on the column strength, why the only reduction left is from the geometrical imperfection.

7. On the subject of an energy approach

An ideal model is simple and accurate. For the present subject at hand with the erratic behaviour of residual stresses, an additional important criteria would be versatility, as to incorporate different residual stress distributions in conjunction with an initial deflection.

In regard to the Lehigh model, it was shown that its simplicity can be argued as compromised. However, it enables the possibility of every imaginable residual stress distribution, provided they can be expressed as a function $f(z_0)$ relating to figure 3.1. As for the Ayrton-Perry model, it has been shown that it is indeed simple to utilize, and in relation to the Lehigh model presents favourable results. However, as the model is based on the assumption of a linear residual stress distribution, its versatility can be argued as compromised. As for the methods proposed by Engesser and Shanley, these were based on a semi-empirical model which is not the focus for the present study.

Because of these considerations, regards were made in investigating the possibilities of establishing a model based on an energy approach. It should be noted, that no definite model were established in the present study. Thus the following sections rather presents the principle of the proposed approach, and ultimately lists the necessary future work that should be carried out.

7.1 Preliminary assumptions

The approach were based on the following preliminary assumptions

- Only small strains are assumed.
- All fibres are assumed to behave as observed in tensile coupon tests, neglecting the Piobert effect and strain hardening, i.e. ideal elastic-plastic behaviour – see figure 7.1.
- Quasi static movement between deformation states.
- Dissipation of strain energy from the straight column configuration to the flexed is neglected.
- The superimposed residual stresses are symmetric around the transverse axis of the variation axis.
- Focus is only placed on the non-linear response of the column as a whole, i.e. above σ_p.
- It is assumed the non-linear behaviour solely arise from residual stresses, i.e. the column is assumed perfectly straight.
- In the linear-elastic region, $\sigma \leq \sigma_p$, residual stresses have no influence.

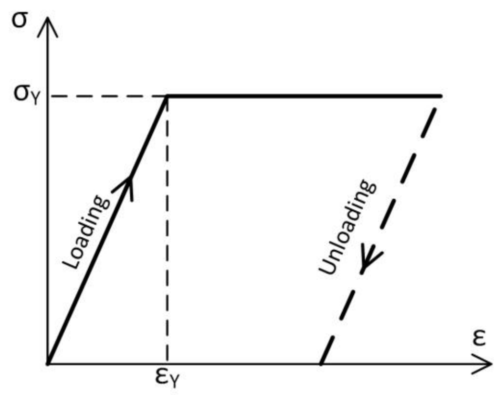

Figure 7.1 Assumed fibre stress-strain response

7.2 Basics of the approach

Consider a closed mechanical system in which only conservative forces act, the total mechanical energy of the system is constant

$$E_{\text{mech}} = E_{\text{pot}} + E_{\text{kin}} = \text{constant} \tag{34}$$

By regarding static systems and assume a quasi-static movement between incremental deformation states, $E_{kin} \approx 0$. Thus, for a closed system, to total potential energy is constant

$$E_{\text{mech}} = E_{\text{pot}} = \text{constant} \tag{35}$$

By regarding elastic deformable bodies, the total potential energy of the system can be expressed as

$$E_{\text{pot}} = U + V \tag{36}$$

Where U is the potential energy stored by the member and dependent on the cross-section and V is the energy potential of the external forces given as $V = -W$, where W is the work.

The approach will be based on the so-called pinned-pinned column system with a wide-flange cross-section – see figure 7.2. It is assumed the buckled shape is governed by

$$w(x) = a \sin\left(\frac{x}{L}\pi\right) \tag{37}$$

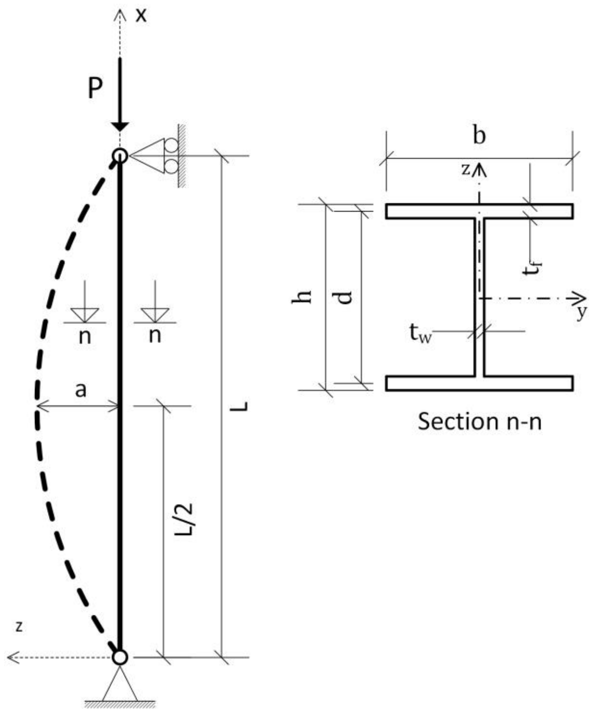

Figure 7.2 Pinned-pinned column system with assumed buckling shape

Principally, the approach can follow two paths. Either the residual stresses have a contribution to the external forces. This approach warrants for a restraint such that their influence only becomes apparent at the onset of non-linear buckling, i.e. when the applied load reaches σ_P. The present study have applied another approach, where the dissipated energy of the system is regarded. By this, the potential energy of the system will not remain constant as stated by equation (35), as some of the energy will dissipate due to yielding of the fibres [10]. Denoting the total dissipated energy of the system by U_{diss}, it is seen that the potential energy can be written as

$$E_{\text{pot}} = U + V - U_{\text{diss}} \tag{38}$$

In order to account for the residual stress distribution in the web and flanges, the approach partitions the cross-section into the top and bottom flange, denoted by index 1 and 2 respectively, and the web denoted by index 3. Thus, the approach seek to determine the dissipated energy from each contributor as $U_{diss,1} + U_{diss,2} + U_{diss,3}$ and thus recalculate the potential energy and minimize it in order to obtain the position of equilibrium.

The potential energy for the system in figure 7.2 with no dissipation can be expressed as

$$E_{\text{pot}} = \frac{EI_y}{2}\int_0^L \left(\frac{\partial^2 w(x)}{\partial x^2}\right)^2 \mathrm{d}x - \frac{P}{2}\int_0^L \left(\frac{\partial w(x)}{\partial x}\right)^2 \mathrm{d}x \tag{39}$$

As such, the problem is reduced to determining the dissipated energy related to the form of the residual stress distribution.

7.3 Dissipation of energy

As the cross-section has been partitioned into three contributors, only a single one will be regarded, as the procedure will be similar for the rest. Thus, the following considerations are to outline the idea of the establishment of an energy model.

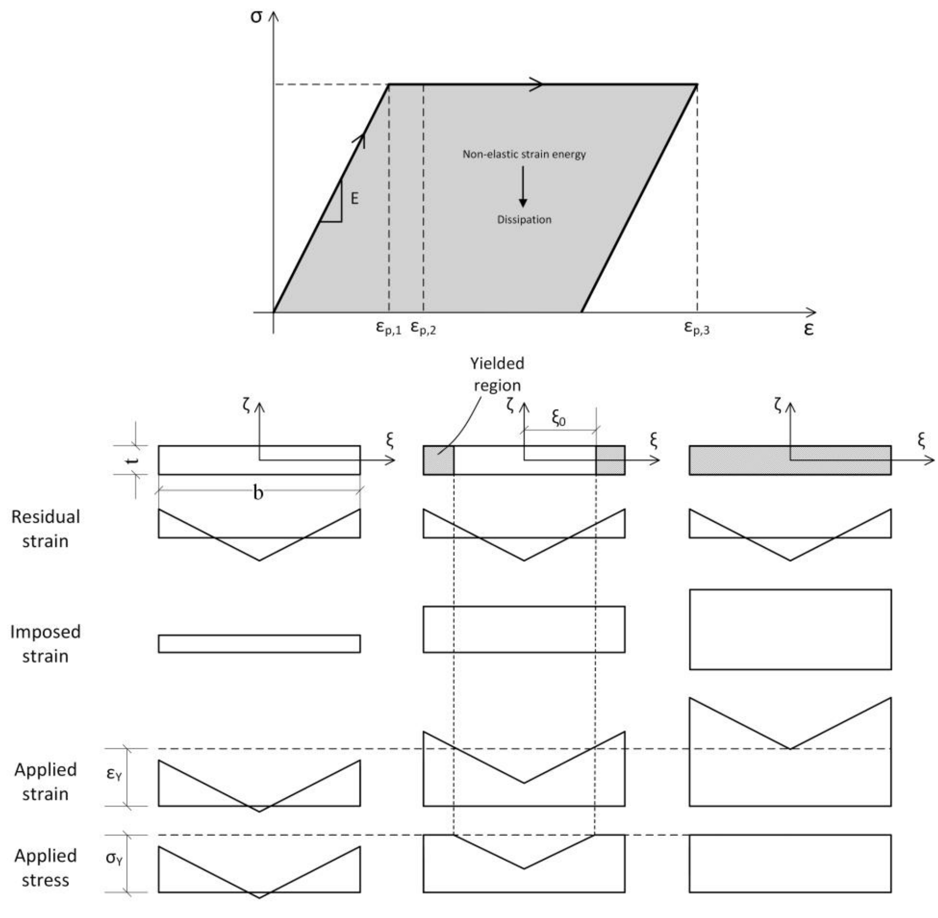

Figure 7.3 Principle in determining the dissipated energy from the partially yielded plate

Considering the small plate in figure 7.3, the various states of the plate is sketched. At the onset of applying an axial load, it will remain elastic and unaffected by residual strains. Increasing the imposed strain will ultimately result in the outermost fibre to yield. As seen, the plate is located in a local Cartesian coordinate system, where the width of the yielded zone is described by the variable *ξ_0*. As such, a general expression for the dissipated energy can be written as

$$U_{\mathrm{diss}} = \int_V \sigma_Y \, \varepsilon(\xi_0, x) \mathrm{d}V = 2t\sigma_Y \int_{\xi_0}^{\frac{b}{2}} \int_0^L \varepsilon(\xi_0, x) \mathrm{d}x \mathrm{d}\xi \tag{40}$$

By this approach, the problem have been limited to the consideration of the strain state for the yielded part of the body.

As of the point of writing no conclusive relation has been established by these considerations.

7.4 Future work

As the residual strain distribution is constant in its configuration, and the imposed strains are assumed uniform. Thus the applied plastic strains will increase according to the inclination of the residual strain distribution for each incremental decrease of ξ_0. However, the dissipated strain energy also depends on the entire volume of the considered body. Thus, a relation between the extents of the plastic strains in the longitudinal direction is necessary to establish. Thus, as indicated in equation (40), the plastic strains will be a function of both x and ξ_0.

Furthermore, as the beam bends into its buckled state, strain reversal will occur at the convex portion of the volume, as in the principle of Engesser's model. Thus, some of the cross-section will regain strain energy, which should further be accounted for.

8. Model discussion

The various models presented in section 3 through section 6 gives rise to a discussion of their results and important aspects observed in each of these. Further, a comparison of the models yielded a further insight into the individual models capabilities and limitations.

8.1 The Lehigh model

The Lehigh model is derived on the assumption of a perfectly straight column, i.e. no geometrical imperfections were assumed present in the column system. As was evident from figures 3.5 and 3.7, the model showed a high variation in the reduction of a columns buckling strength due to the presence of residual stresses. This reduction is highly dependent on what assumed residual stress distribution is considered. Generally, those usually applied are the linear and parabolic distribution, as noted by Jönsson and Stan [13]. Figures 8.1 and 8.2 show their range of influence on the buckling strength of a column. As is evident, the range of these two types of distributions approximately constitute as the average of all eight assumed residual stress distributions. Notably is, that the linear distribution gives the highest reduction in the buckling strength for this range. This could substantiate why many models implement the influence of residual stresses based on a linear distribution.

During the comprehensive investigation of residual stresses carried out at Lehigh University, a large number of residual stress distributions were obtained experimentally, see e.g. Fujita [9], Huber and Beedle [12]. Many of these corresponded well with the range prescribed by the linear and parabolic distribution in Figures 8.1 and 8.2. However, some deviated significantly from these, where distributions were observed that would decrease the buckling strength below the linear distribution. As is evident from equation (2), it is the residual stresses over the entire cross-section that need to equilibrate. Thus, if in the manufacturing process a detrimental differential cooling takes place, the entire web could be in a state of tension, thus requiring a high amount of compressive stress in the flanges. This tendency was also observed in the measurements from Lehigh.

Furthermore, by comparing the reduction in buckling strength for buckling around the major and minor axis, see figure 8.3, it is evident that the weak axis is severely reduced. This tendency corresponds well with the rate at which the yielded part of the cross-section happens. About the major axis, the reduction of the buckling strength happens at a slower rate due to the fact that the mass furthest from the neutral axis yield slower, thus maintaining more of its flexural stiffness. The vice versa is the case for buckling around the minor axis.

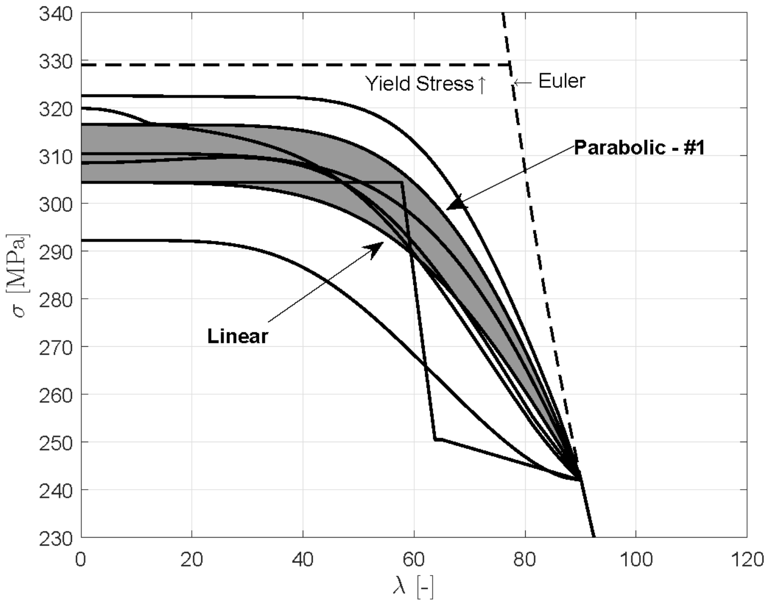

Figure 8.1 Column-curve for the strong-axis of an HE300B ross-section with a field of the affected area on the curve

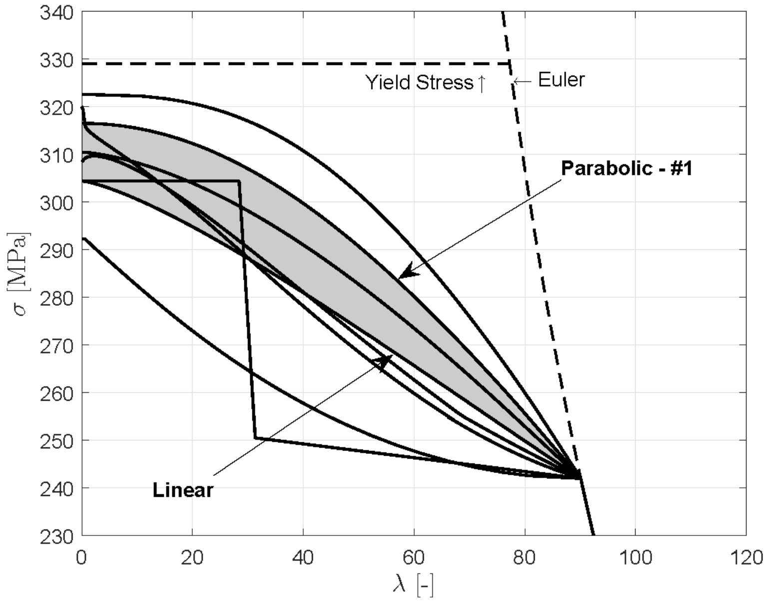

Figure 8.2 Column-curve for the weak-axis of an HE300B cross-section with a field of the affected area on the curve

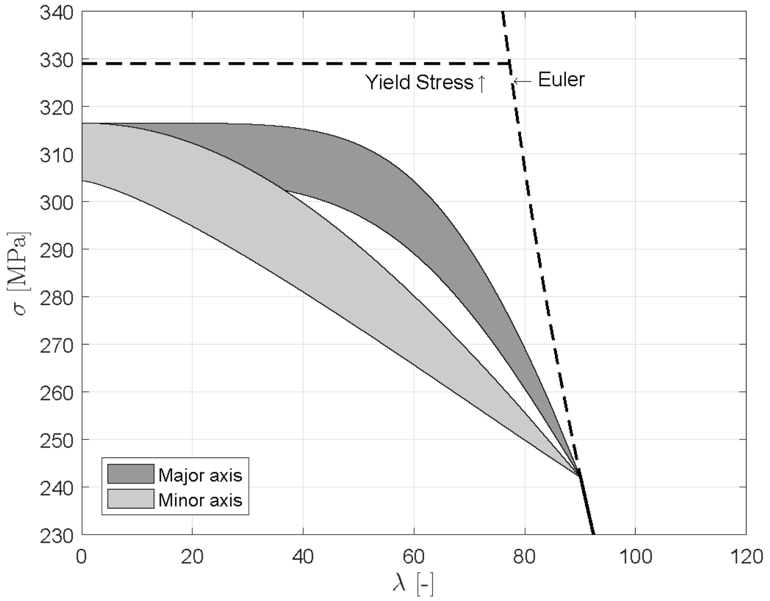

Figure 8.3 Comparison of the reduction of buckling strength around the major and minor axis

8.2 The Ayrton-Perry model

Referring to figure 8.4 and 8.5, where the reduction in buckling strength, as proposed by the Ayrton-Perry model for buckling curves *a* and *b*, are compared with the Lehigh model for the range between a linear and parabolic residual stress distribution. Not much correlation is observed between these. This is due to their initial assumptions. Initially, the Lehigh model does not consider the effects of geometrical imperfections, and as such, the onset of the reduction of a columns buckling strength, corresponds with the proportionality limit. However, as the Ayrton-Perry model equates the residual stress distribution as an geometrical imperfection, their influence is also accounted for in the linear buckling region. As stated by Jönsson and Stan [13], a column with only geometrical imperfections and no residual stress can be expressed by an initial imperfection of *L/1000*. As the buckling curves *a* and *b* have α ranging from 0.21 to 0.34, corresponding to an initial deflection ranging from *L/300* to *L/200*, the contribution from equating the residual stresses as geometrical imperfections has a high influence prior to the onset of non-linear buckling, thus amplifying the buckling strength reduction in the entire non-linear region.

If the Ayrton-Perry model for an initial geometrical imperfection of *L/1000* is compared to the Ayrton-Perry model for the range of *a* and *b* buckling curves, the reduction in buckling strength due to equating residual stresses as geometrical imperfections is apparent – see figure 8.6.

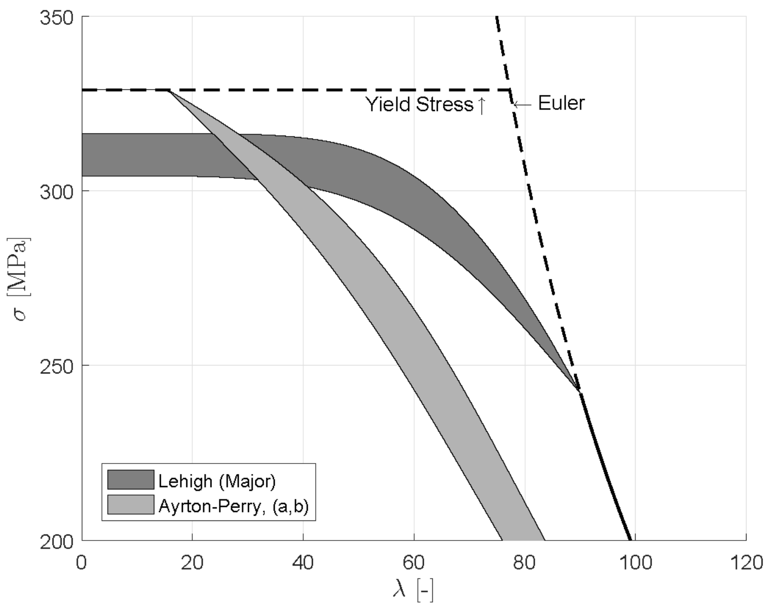

Figure 8.4 A comparison of the Lehigh model for buckling around the major axis and the Ayrton-Perry model for buckling curves *a* and *b*

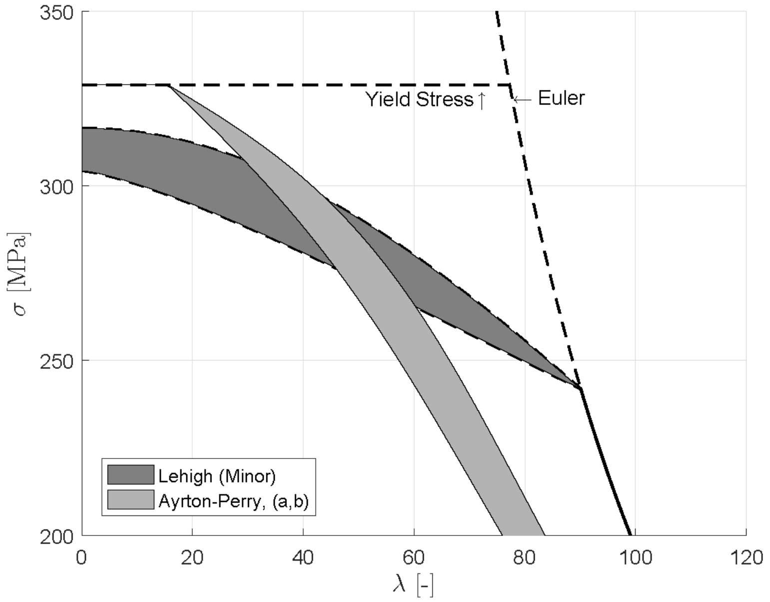

Figure 8.5 A comparison of the Lehigh model for buckling around the minor axis and the Ayrton-Perry model for buckling curves *a* and *b*

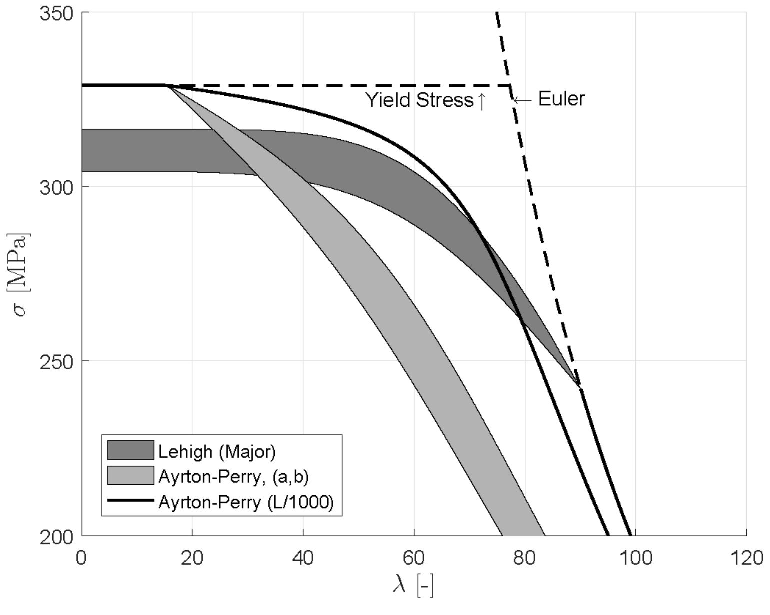

Figure 8.6 A comparison of the Ayrton-Perry model for buckling curves ranging from *a* to *b* and for the case of only imperfections

8.3 The modified Lehigh model

By comparing the modified Lehigh-model with the Ayrton-Perry model, for *a,b*-curves, it is seen that a complete overlaps is yet to be obtained, which is evident from figure 8.7. However, parallels is observed. Due to the fact, that the modified Lehigh model is only defined for non-linear buckling it is not capable of accounting for the influence of geometrical imperfections in the region of linear buckling. However, as is evident, the onset of non-linear buckling has been shifted to a lower critical stress level by a factor σ_m. Furthermore, the inclination of the modified Lehigh model seem to follow that of the Ayrton-Perry model slightly more than the Lehigh model. Thus, if the modified Lehigh model were capable of fully accounting for the influence of geometrical imperfections in the linear region, the two models could have a better fit than that observed in the present. In order to see if a change in the imperfection value of *L/1000* would change the value of the modified Lehigh model, it was changed to range between *L/600 – L/1000*, which is seen in figure 8.8. It is seen in the plot that the value converges towards the values from the Ayrton-Perry model when the value of geometrical imperfections goes towards *L/600*. However, it is also given, that for lower values than this, the calculations cannot be completed using a maximum of 600 iterations for each calculation. This is a direct consequence of the somewhat more complex model set up in order to account for the skew rate of yielding of the cross-section.

However, it should be noted that values solely accounting for geometrical imperfections of *L/600* is well above the value suggested in DS/EN-1993 [4].

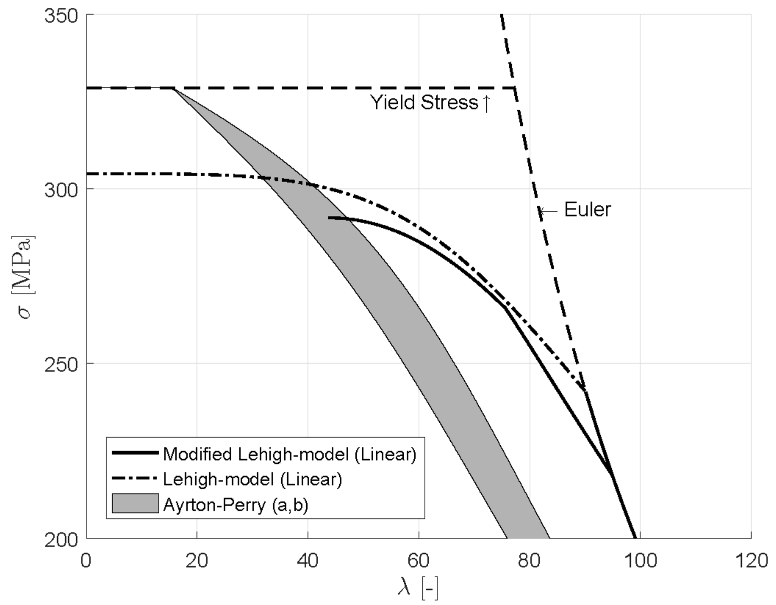

Figure 8.7 Comparison between Lehigh, the modified Lehigh model and the original Ayrton-Perry model

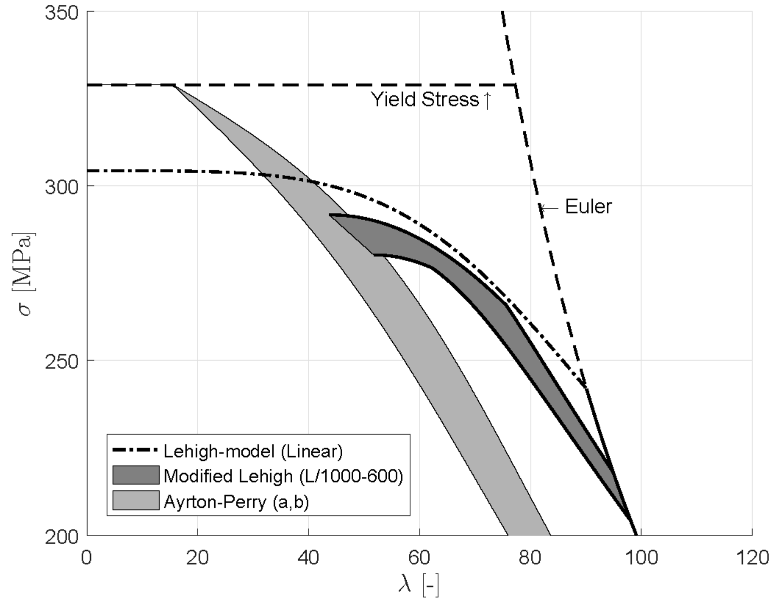

Figure 8.8 Comparison of Ayrton-Perry and a range of the modified Lehigh model with geometrical imperfections ranging from *L/1000* to *L/600*

8.4 The modified Ayrton-Perry model

The modified Ayrton-Perry model is not a model in its own right. It is a combination of two models. The Lehigh models strength is given in accurate predictions for the reduction of the buckling strength due to residual stresses, whereas the Ayrton-Perry model accurately accounts for the influence of initial geometrical imperfections. Thus, a combination is presented in figure 8.9.

As is evident, there is a correlation between the modified Ayrton-Perry model with the original Ayrton-Perry model for the *a,b*-curve. This is especially given within the range of reduction in regard to the yield stress, until point 2. Afterwards, the reduction from both the Lehigh-model and the imperfection keeps increasing. However, due to the Euler-curve decreasing even more, the numerical value of decrease from the two models actually gives a small increase from the end point of before. So, it should not be seen as an increase in capacity due to a higher slenderness ratio, but instead an increase due to the value to which the reduction happens. The slope for the Euler-column is steeper than that of the two functions, why a reduction is not able to be as large in value. If we regard three numerical points between point 2 and 3 these indicate the reason for this behavior of the model, see table 8.1. Thus, the increase in buckling strength between point 2 and 3 in figure 8.9 is due to the method of which the models are conjoined. It is seen in Alpsten [1] that the Ayrton-Perry model is corroborated by experimental data. Further the model from Lehigh is also corroborated by their own sub column experiment, as seen in figure 23-30 in Huber and Beedle [12]. Nonetheless, the modified Ayrton-Perry model combines two different methods that are based on different assumptions and take different approaches in establishing the reduction of a columns buckling strength. Although the Ayrton-Perry approach considers the influence of residual stresses, it does so without regard to their actual influence, which is evident in the linear buckling region. This is supported by experimental data presented in ECCS [5]. Figure 3.1.1.7-.10 and 3.1.2.2-.5 indicates that for large values of λ, i.e. slender columns, the critical buckling strength is higher than that predicted by the Ayrton-Perry model. Thus, it could be argued that applying the Ayrton-Perry model with an initial geometrical imperfection of *L/1000* for columns in the linear buckling region could be valid. However, this statement is unsubstantiated by clear experimental data.

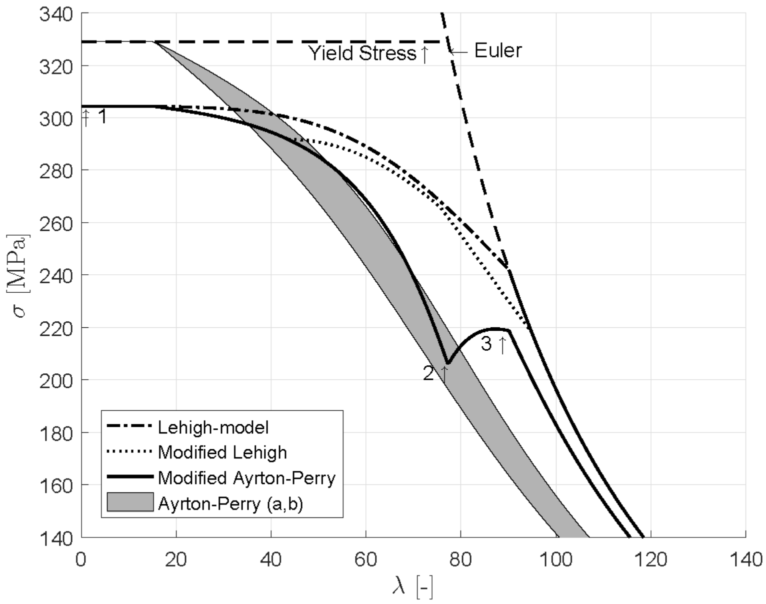

Figure 8.9 Comparison of the Ayrton-Perry and the modified Ayrton-Perry model

Table 8.1 Overview of three different slenderness ratios and their corresponding values of stress between point 2 and 3 in figure 8.9

λ	**σ_{rs}**	**σ_{imp}**	**σ**	**Δσ**	**σ_{mod}**
[-]			[MPa]		
77.16	265.53	269.83	329.00	122.63	206.37
81.63	257.72	252.86	294.78	78.97	215.80
90.10	241.92	218.73	241.92	23.17	218.73

8.5 General discussion

As was evident from the previous sections, the models yielded widely different results. However, another important factor that differs for the models were their simplicity in relation to their accuracy, and also their versatility. One of the key features of a model is that it needs to be as simple as possible, without presenting erroneous results. As such, assessing the individual models simplicity in relation to their accuracy is good common practice.

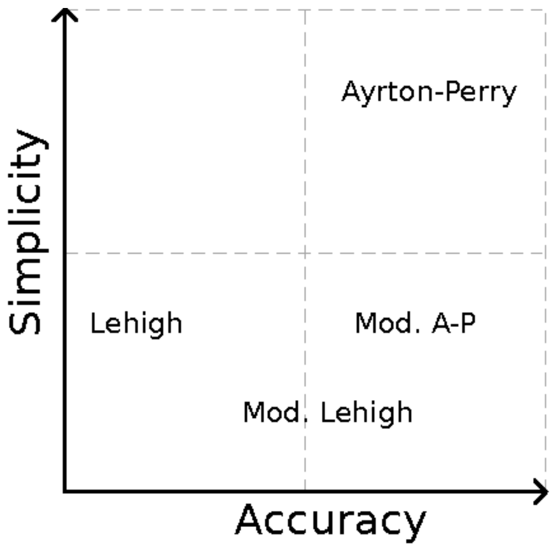

Figure 8.10 A principal sketch of a simplicity and accuracy diagram for the models

Thus, in accordance with this, figure 8.10 presents a diagram in which the individual models simplicity in relation to their accuracy is assessed. In the figure is given an estimate on where the different models should be placed. Where it is given that, due to the Ayrton-Perry model being used by the European Standard along with a well-based experimental background, the model is given as a very simple and accurate model. The other models are therefore placed relative to their simplicity and accuracy compared to the Ayrton-Perry model. As described earlier, the Lehigh model is also well-based by previous experimental data. However, since the Lehigh-model is given by comprehensive calculations, its simplicity is low relative to the Ayrton-Perry model, which only requires the calculation of a column reduction factor, χ, as described in section 4. Instead, the Lehigh model is seen to be relatively more versatile, since it allows for a change in magnitude and distribution of the residual stress, where the Ayrton-Perry model is only given for a single magnitude and distribution. For the modified models it is given that the accuracy of the modified Lehigh is increased relative to the original model, since it incorporates the initial deflection. However, this also complicates the model severely. The modified Ayrton-Perry is as relatively accurate as the original Ayrton-Perry in the principal sketch, this is however not possible to determine without extensive experimental work and results, that could substantiate the model and its assumptions.

9. Concluding remarks

A genera conclusion, is that the non-linear buckling behaviour of a column is highly troublesome to accurately predict through general analytical models. This is especially due to the erratic nature of residual stresses. As is evident from the article, the generalized Lehigh model is highly versatile in accounting for the arbitrary distribution of residual stress. If the required distribution of residual stress can be express as a function, the Lehigh model can account for its reduction of the columns buckling strength. As such, the generalization and versatility of the Lehigh model is verified through the use of the eight different types of residual stress distributions in figure 3.3. Further, the results for the reductions in the buckling strength from this method complies with a physical regard of the non-linear behaviour of the material, as is evident in figures 3.5, 3.7 and 8.3.

In regard to the Ayrton-Perry model, it is evident that its use in the European Standard DS/EN-1993 [4] is based on its highly simple nature, and due to it being corroborated by experimental comparisons. However, in regard to the influence of residual stress, the models is highly limited. Neither the distribution nor the magnitude of the residual stresses can be altered in the model. It is based on predetermined values for the magnitude of the residual stress and a linear distribution of these. The justification for the linear distribution can be observed in figures 8.1 and 8.2. Here, the Lehigh model shows that the linear distribution is based on a somewhat conservative estimation, in relation to distributions obtained experimentally. However, the model does not account for the actual influence that residual stresses impose on a column. This is due to them being equated as an geometrical imperfection, where an initial imperfection of *L/1000* is only due to imperfections and *L/300* to *L/250* is both due to imperfection and the residual stress.

In order to make the Lehigh model more accurate, this was modified in order to account for geometrical imperfections. This was achieved by superimposing a bending stress with the residual stress distributions. The model presented a higher reduction of the columns buckling strength, as is observed in figure 8.8. However, the reduction was not of the same magnitude as the Ayrton-Perry model. This is due to the model only regarding the non-linear buckling behaviour. As it is based on obtaining a critical stress by axial equilibrium, and relating this to a modification of the Euler formula given as

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \frac{I_e}{I} \tag{41}$$

Thus, when the cross-section becomes fully elastic, the critical stress level is based on the original Euler formula as $I_e/I = 1$. Furthermore, as the model introduces a skew rate of which the cross-section yields, the model becomes highly complex with a somewhat limited usage.

Further, as the Lehigh model is highly accurate in accounting for the influence of residual stress, however lacks a proper implementation of the influence of initial geometrical imperfections. In contrary, the Ayrton-Perry model is highly accurate in accounting for the influence of initial geometrical imperfection, but not on the implementation of the influence of residual stress. Thus, the results from these models were combined in order to observe the combined reduction in buckling strength. As seen in figure 8.9, this approach showed interesting properties compared to the original Ayrton-Perry model. However, as the approach was a combination of results obtained from models with different considerations of the problem, no final conclusions on a columns behavior can be drawn from this result.

Ultimately, an investigation of the possibilities in establishing a model based on the energy principles were carried out. The model approached the problem by regarding the gradually dissipating energy, as the cross-section becomes increasingly plastic. However, the connection between the incrementally developing area of the yielded cross-section and the dissipated energy is not established. Thus, no final model is presented in the current report.

9.1 Further studies

As described earlier, the lack of experimental results from stub column tests, withholds the article from committing to a validation on the modified models and their authenticity with regards to true values. Therefore, in order for a well-based argumentation to either disregard or approve the models, a thorough experimental work is needed. Such work could consist of around 30 stub column test, divided into three subgroups, where each group would give a more detailed investigation of a specific point on the column stability diagram. With regards to figure 6.1, specific ranges of interest would be for low values of λ, the transition range in which residual stress start to influence the column strength, and finally in the range for slender columns.

Additionally, further work on establishing a model on the energy principles could possibly lead to a more versatile, simple and accurate model than those presented here. This would be the case if the model is able to account for both the influences of an arbitrary residual stress distribution and geometrical imperfections under the same principles, for both linear and non-linear buckling. With a successful energy-model, it would be possible for structural engineers to design columns to a specific case, thereby allowing for accurate calculations instead of mere approximations, as is done by many contemporary models.

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