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Required anchorage length of reinforcement in circular foundation subjected by a centrally located normal force and supported by a uniformly distributed reaction

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Introduction

In this article, the maximum anchorage length for the reinforcement in the bottom of a circular point foundation is determined to avoid having to use bent-up reinforcement. The work presented in this article is a continuation of the work presented in [1], where an exact solution is presented for a point foundation affected by a column reaction of finite extent.

When determining the maximum anchorage length for foundations affected by moments, it is assumed that the foundation is provided with orthotropic reinforcement with a constant degree of reinforcement in all sections. First, the required anchorage length for the reinforcement in a point foundation affected by centrally located normal force acting as a point load is examined. Next, the necessary anchorage length for a point foundation affected by a centrally located normal force acting over a finite area is examined.

The fact that the anchoring of the reinforcement is modeled as linearly growing from the end of the reinforcement means that the foundation will appear anisotropically reinforced near the edge. This matter is also addressed.

Point foundation subjected bya centrally located normal force acting as a point load

Cf. [1] the moment distribution in a circular foundation affected by a point load can be expressed by:

$$m_r = 0 \tag{1}$$

$$m_\theta = \frac{P}{2\pi}\left(1 - \left(\frac{r}{a}\right)^2\right) \tag{2}$$

$$m_{r\theta} = 0 \tag{3}$$

[1] Aarhus University, Value Engineering ApS

The point foundation subjected by a point load, P, is shown in Figure 1.

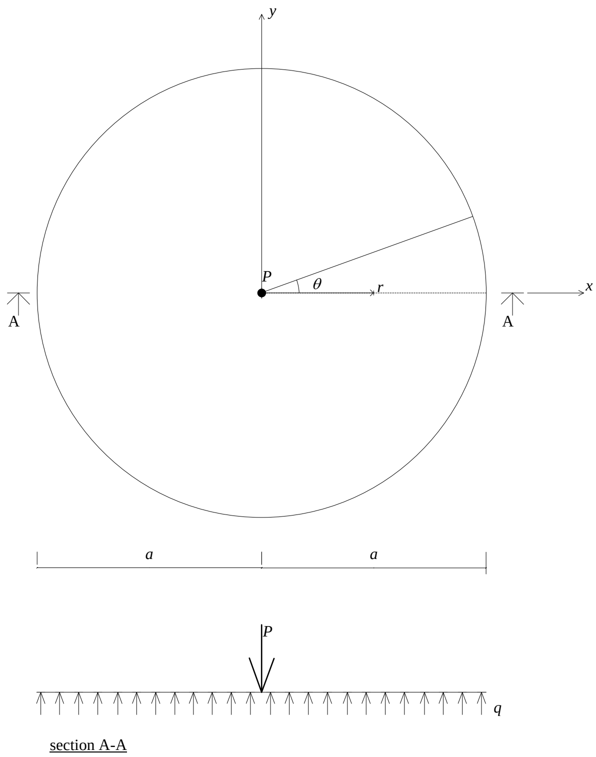


Figure 1 Circular foundation subjected by a point load

Since the study is about assessing the necessary anchoring capacity with homogeneous, orthogonally placed reinforcement at the bottom of the foundation, the moments are rewritten as m_x, m_y and m_{xy}, see appendix A:

$$m_x = m_r \cdot \cos^2\theta + m_\theta \cdot \sin^2\theta + 2m_{r\theta} \cdot \sin\theta \cdot \cos\theta \quad (4)$$

$$m_y = m_r \cdot \sin^2\theta + m_\theta \cdot \cos^2\theta - 2m_{r\theta} \cdot \sin\theta \cdot \cos\theta \quad (5)$$

$$m_{xy} = m_r \cdot \sin\theta \cdot \cos\theta - m_\theta \cdot \sin\theta \cdot \cos\theta + m_{r\theta}(\sin^2\theta - \cos^2\theta) \quad (6)$$

With (1) - (3) inserted in (4) - (6) we have:

$$m_x = \frac{P}{2\pi}\left(1 - \left(\frac{r}{a}\right)^2\right) \cdot \sin^2\theta \quad (7)$$

$$m_y = \frac{P}{2\pi}\left(1 - \left(\frac{r}{a}\right)^2\right) \cdot \cos^2\theta \quad (8)$$

$$m_{xy} = -\frac{P}{2\pi}\left(1 - \left(\frac{r}{a}\right)^2\right) \cdot \sin\theta \cdot \cos\theta \quad (9)$$

Since for $x \geq 0$ and $y \geq 0$ it holds that:

$$\theta = \text{atan}\left(\frac{y}{x}\right) \quad (10)$$

$$r = \sqrt{x^2 + y^2} \quad (11)$$

We have by insertion in (7) - (9):

$$m_x = \frac{P}{2\pi}\left(1 - \frac{x^2+y^2}{a^2}\right) \cdot \sin^2\left(\text{atan}\left(\frac{y}{x}\right)\right) \quad (12)$$

$$m_y = \frac{P}{2\pi}\left(1 - \frac{x^2+y^2}{a^2}\right) \cdot \cos^2\left(\mathrm{atan}\left(\frac{y}{x}\right)\right) \tag{13}$$

$$m_{xy} = -\frac{P}{2\pi}\left(1 - \frac{x^2+y^2}{a^2}\right) \cdot \sin\left(\mathrm{atan}\left(\frac{y}{x}\right)\right) \cdot \cos\left(\mathrm{atan}\left(\frac{y}{x}\right)\right) \tag{14}$$

The principles for dimensional moments m_{sx} and m_{sy} in the direction of the *x*- and *y*-axis, respectively, are given by [2]:

$$m_{sx} = m_x + \frac{1}{\chi}\left|m_{xy}\right| \tag{15}$$

$$m_{sy} = m_y + \chi\left|m_{xy}\right| \tag{16}$$

Where χ expresses the slope of the inclined concrete pressure, θ_c, from the torsional moment, that is $\chi = \cot\theta_c$.

Since the foundation is equally reinforced in the *x* and *y* directions, the most optimal utilization is obtained by having the design moment be the same in the two directions. This can be achieved by using a value of χ that ensures this:

$$m_{sx} = m_{sy} \tag{17}$$

With expressions for m_{sx} and m_{sy} inserted:

$$m_x + \frac{1}{\chi}\left|m_{xy}\right| = m_y + \chi\left|m_{xy}\right| \tag{18}$$

From this expression, χ can be determined:

$$0 = \chi^2\left|m_{xy}\right| + \left(m_y - m_x\right) \cdot \chi - \left|m_{xy}\right| \tag{19}$$

$$0 = \chi^2 + \frac{(m_y - m_x)}{|m_{xy}|} \cdot \chi - 1 \tag{20}$$

That is:

$$\chi = -\frac{1}{2}\frac{(m_y - m_x)}{|m_{xy}|} + \frac{1}{2}\sqrt{\frac{(m_y - m_x)^2}{m_{xy}^2} + 4} \tag{21}$$

With expressions for m_x, m_y and m_{xy} inserted:

$$\chi = -\frac{1}{2}\frac{(\cos^2(\theta) - \sin^2(\theta))}{\sin(\theta)\cdot\cos(\theta)} + \frac{1}{2}\sqrt{\frac{(\cos^2(\theta) - \sin^2(\theta))^2}{(\sin(\theta)\cdot\cos(\theta))^2} + 4} \tag{22}$$

The projecting factor in (12) - (14) is evaluated:

$$\omega_x = \sin^2\left(\mathrm{atan}\left(\frac{y}{x}\right)\right) \tag{23}$$

$$\omega_y = \cos^2\left(\mathrm{atan}\left(\frac{y}{x}\right)\right) \tag{24}$$

$$\omega_{xy} = \sin\left(\mathrm{atan}\left(\frac{y}{x}\right)\right) \cdot \cos\left(\mathrm{atan}\left(\frac{y}{x}\right)\right) \tag{25}$$

for $\theta = \mathrm{atan}\left(\frac{y}{x}\right)$.

The variation is the same in the *x* and *y* directions respectively. Based on (17), the most critical angle for m_{sy} is found in the interval $0 \leq \theta \leq \pi/4$.

The moment is therefore greatest for $\theta \to 0$, where m_x and m_y acts alone.

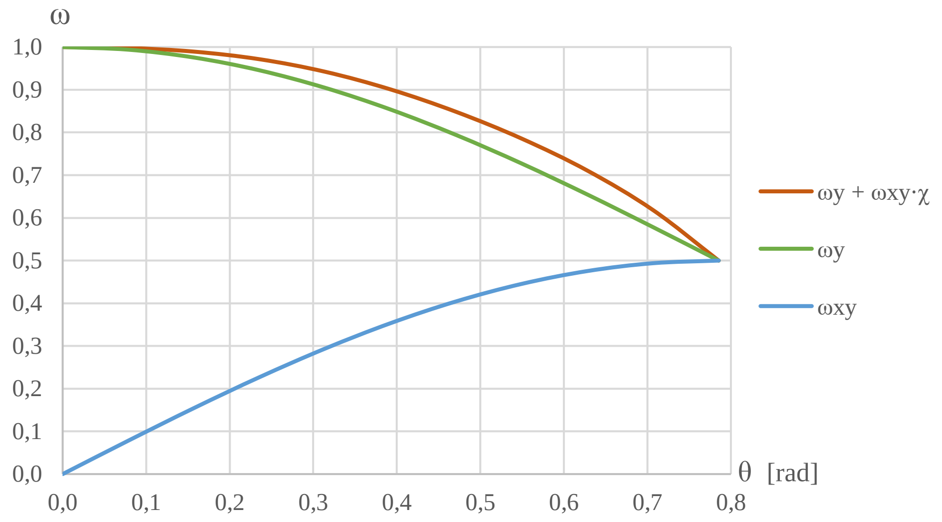


Figure 2. Examination of the variation of m_x, m_y and m_{xy} as a function of θ

In the *x* direction, the variation for m_{sx} is shown in figure 3 with functions for *y/a* from 0.1 – 0.9.

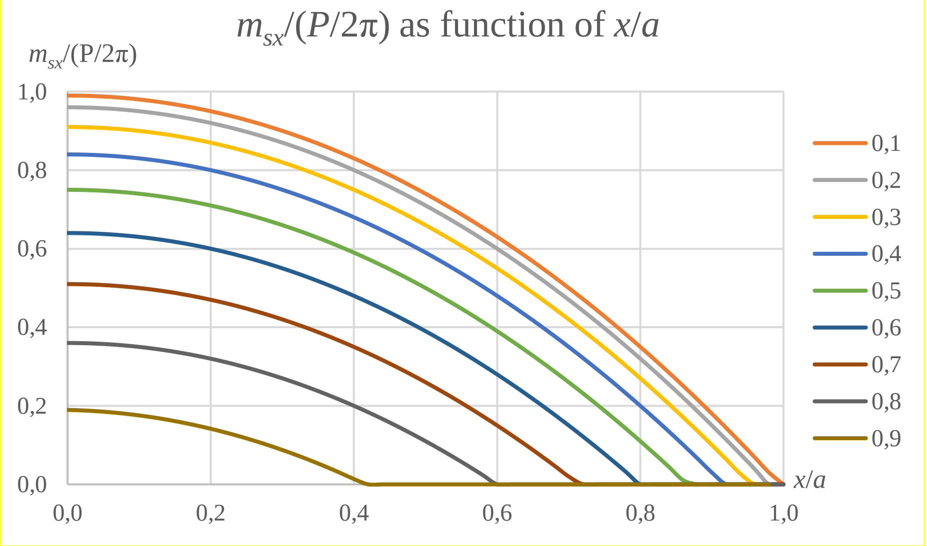


Figure 3 m_{sx} as a function of x/a for different values of y/a

The most critical in terms of anchoring is seen for $\theta \rightarrow 0$, as shown in Figure 2, and at the edge for $x \rightarrow$ a, as shown in Figure 3.

With the expression for χ (22) inserted into the expression for m_{sy} (16):

$$m_{sy} = m_y + \chi|m_{xy}| = \frac{P}{2\pi}\left(1 - \frac{x^2+y^2}{a^2}\right) \cdot \left(\cos^2(\theta) - \left(\frac{1}{2}\frac{(\cos^2(\theta)-\sin^2(\theta))}{\sin(\theta)\cdot\cos(\theta)} + \frac{1}{2}\sqrt{\frac{(\cos^2(\theta)-\sin^2(\theta))^2}{(\sin(\theta)\cdot\cos(\theta))^2} + 4}\right) \cdot \sin(\theta) \cdot \cos(\theta)\right) \tag{26}$$

This can be reduced to:

$$m_{sy} = \frac{P}{2\pi}\left(1 - \frac{x^2+y^2}{a^2}\right) \tag{27}$$

Analog expression is available for m_{sx}:

$$m_{sx} = \frac{P}{2\pi}\left(1 - \frac{x^2+y^2}{a^2}\right) \tag{28}$$

The variation is seen to be identical in the *x* and *y* directions.

The increments in the *y*-axis direction of m_{sy} is given by:

$$\frac{dm_{sy}}{dy} = -\frac{Py}{\pi a^2} \tag{29}$$

The biggest increase is seen to be for $y = a$:

$$\left(\frac{dm_{sy}}{dy}\right)_{max} = -\frac{P}{\pi a} \tag{30}$$

With a constant internal moment arm, *z*, a necessary anchoring capacity per length unit on:

$$\left(\frac{dT}{dx}\right)_{max} = \frac{P}{\pi \cdot a \cdot z} \tag{31}$$

The maximum moment gives a tensile force of:

$$T_{max} = \frac{m_\theta}{z} = \frac{P}{2\pi \cdot z} \tag{32}$$

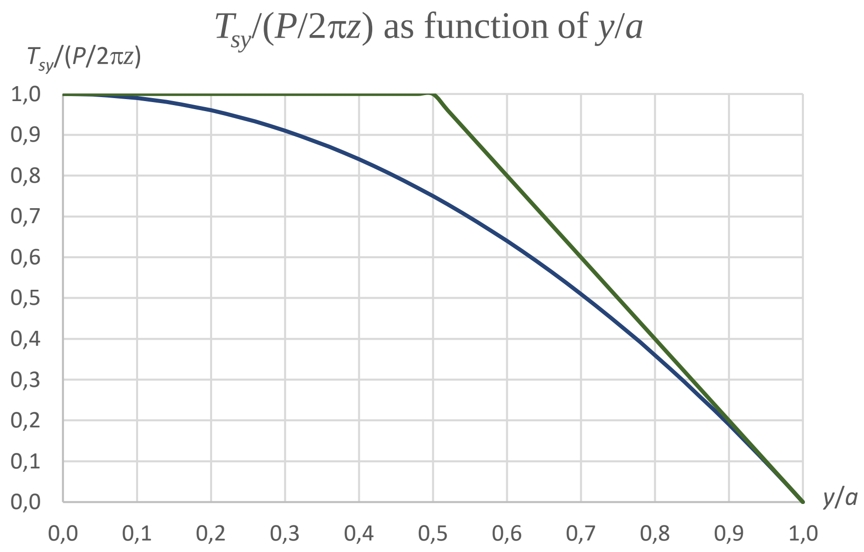
The distance from the edge, $l_{b,max}$, where full capacity must be built up with the necessary increment described by $\left(\frac{dT}{dx}\right)_{max}$, assuming a linear build-up of the force in the reinforcement

$$T_{max} = l_{b,max} \cdot \left(\frac{dT}{dx}\right)_{max} \tag{33}$$

By equalizing (32) and (33) we have:

$$l_{b,max} = \frac{\frac{P}{2\pi \cdot z}}{\frac{P}{\pi \cdot a \cdot z}} = \frac{1}{2}a \tag{34}$$

This result means that the anchorage length must be less than or equal to half the radius of the foundation for a point foundation subjected to a point load.



Figur 4 Illustration of the maximum anchorage length

As can be seen, the distance is independent of the magnitude of the maximum moment.

Point foundation subjected by centrally located column with finite area

Figure 5 shows a point foundation with radius, *a*, that is subjected by a load from a column with a diameter of *D*.

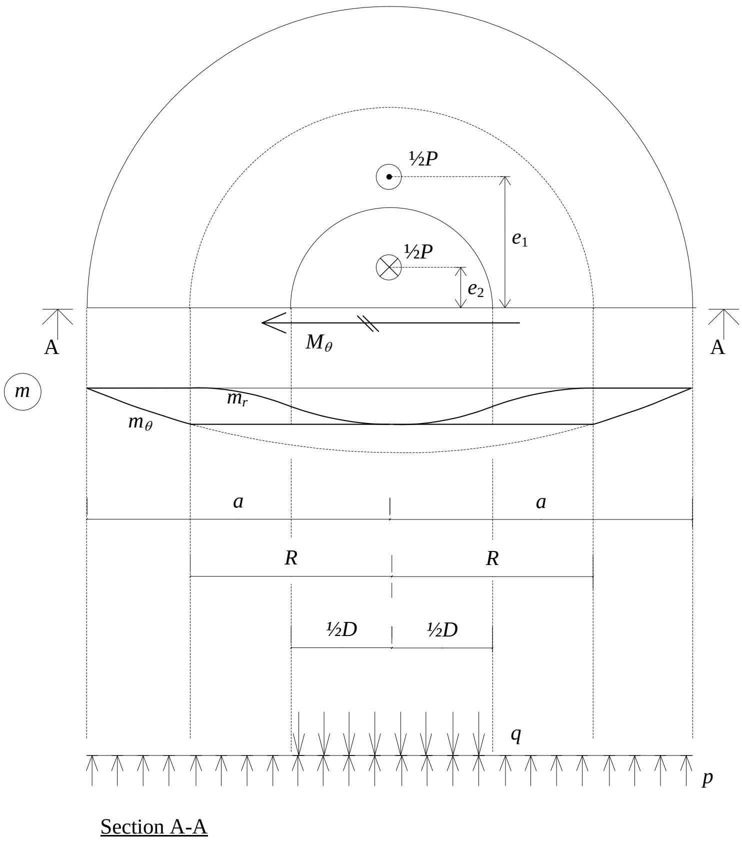


Figure 5 Point foundation with radius a that is affected by a column with a diameter of D

The maximum moment is determined cf. [1] by:

$$m_\theta = \frac{P}{2\pi}\left(1 - \sqrt[3]{\frac{D^2}{4a^2}}\right) \tag{35}$$

Again, the maximum allowable anchorage length can be determined by:

$$T_{max} = l_{b,max} \cdot \left(\frac{dT}{dx}\right)_{max} \tag{36}$$

The increase at the edge is identical to the case where the foundation is considered to be affected by a point load, i.e:

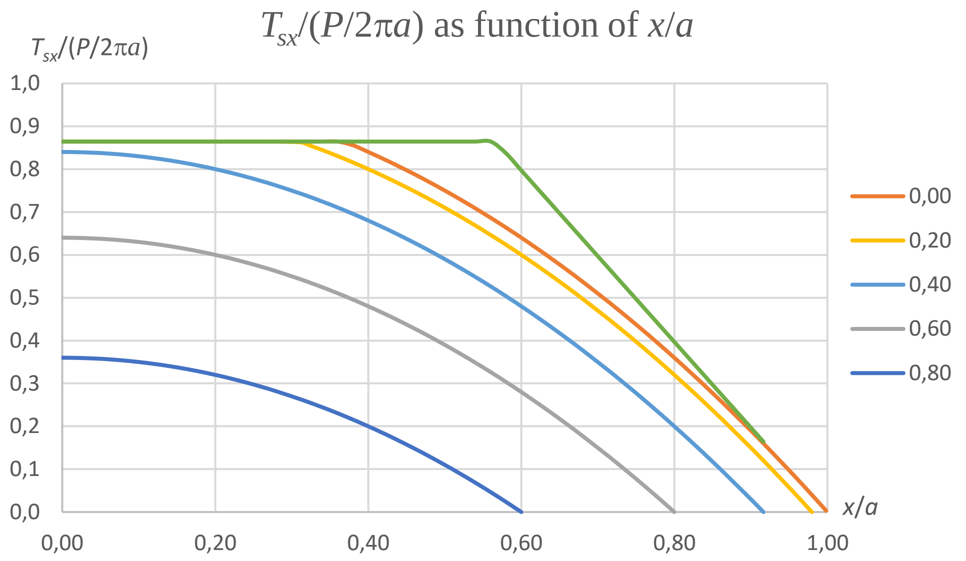
$$\left(\frac{dT}{dx}\right)_{max} = \frac{P}{\pi \cdot a \cdot z} \tag{37}$$

T_{max} is again determined based on the moment:

$$T_{max} = \frac{m_\theta}{z} = \frac{P}{2\pi \cdot z}\left(1 - \sqrt[3]{\frac{D^2}{4a^2}}\right) \tag{38}$$

Maximum anchoring length to avoid bent reinforcement:

$$l_{b,max} = \frac{T_{max}}{\left(\frac{dT}{dx}\right)_{max}} = \frac{1}{2}\left(1 - \sqrt[3]{\frac{D^2}{4a^2}}\right)a \tag{39}$$



Figur 5. $T_{sx}/(P/2\pi a)$ as function of x/a, shown for D/a = 0,1

For later comparison, the maximum anchorage length is shown specifically for y/a = 0.4 based on the same approach with a slope of 2*x*/*a*, see figure 6.

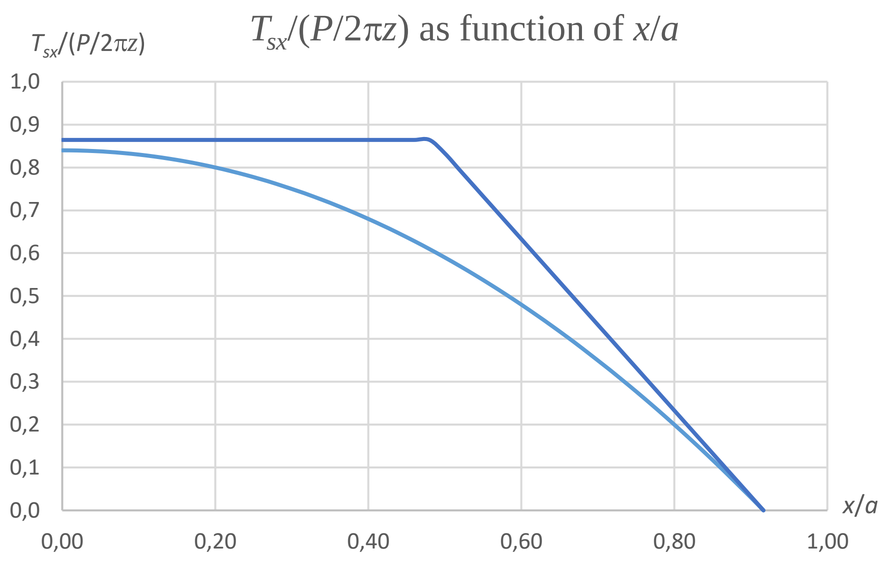


Figure 6 $T_{sx}/(P/2\pi a)$ as function of x/a, shown for D/a = 0,1 and y/a = 0.4

If it is assumed that anchored/bent-up reinforcement is not used at the edge of the point foundation, the moment capacity decreases towards the edge. Since the anchoring is assumed to vary linearly, it is correspondingly assumed that the moment capacity decreases linearly at a distance from the edge corresponding to the anchoring length in both directions.

This is illustrated in Figure 7. A point C is considered which is situated at a distance from the edge which is less than the anchoring length. Since the distance from the edge is different in the two directions, this also means that the moment capacity at this point is different in the two directions.

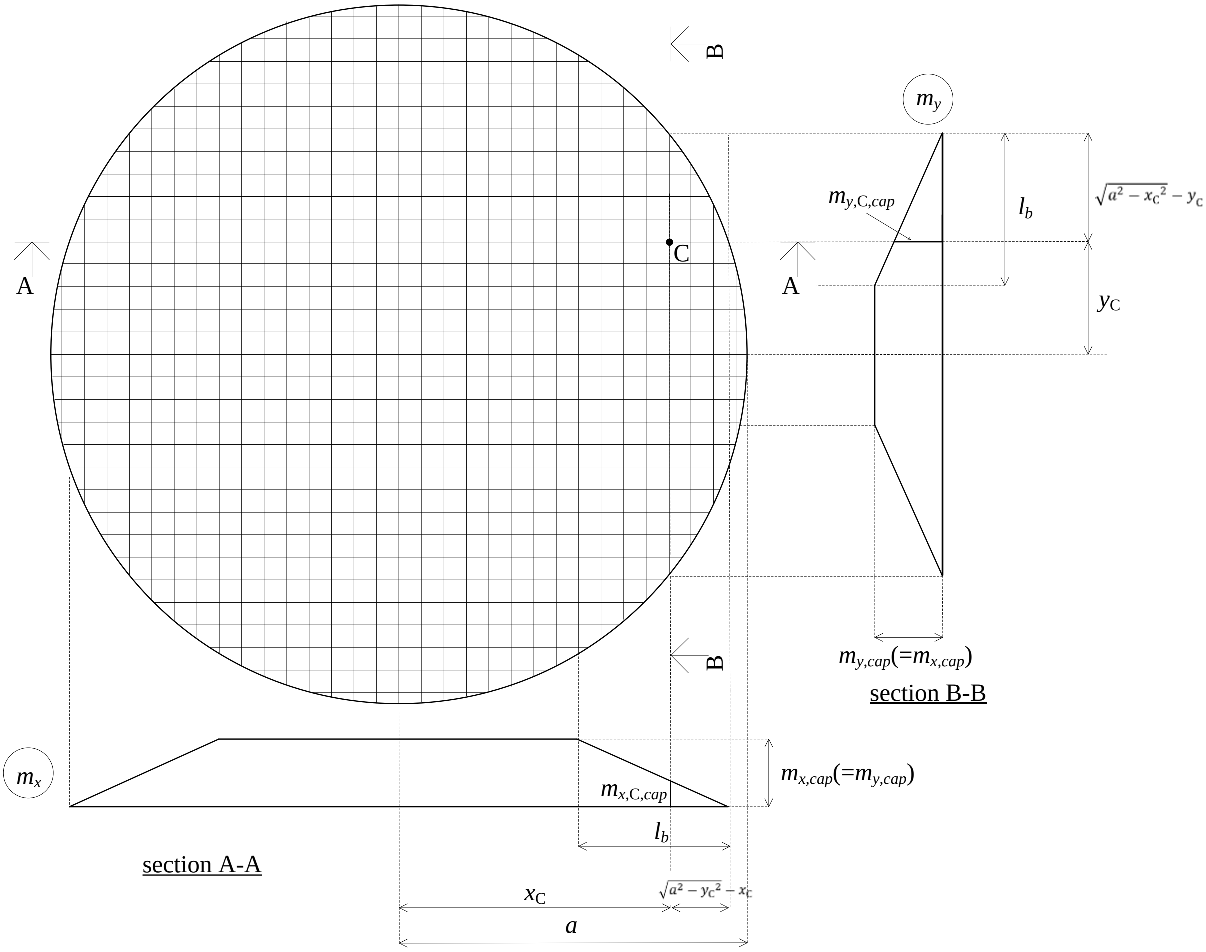


Figure 7 Circular foundation with orthogonal reinforcement and decreasing moment capacity at the edge

The optimal utilization of the reinforcement in a zone closer than the anchorage length from the edge is achieved by taking into account the varying moment capacity. The relationship between the moment capacities in the two directions can be expressed by the inverse of the relationship between the distances to the edge in the two directions. (17) is thus changed to:

$$m_{sy} = \frac{\sqrt{a^2-x^2}-y}{\sqrt{a^2-y^2}-x} m_{sx} \qquad (40)$$

With expressions for m_{sx} and m_{sy} inserted:

$$m_y + \chi|m_{xy}| = \frac{\sqrt{a^2-x^2}-y}{\sqrt{a^2-y^2}-x} m_x + \frac{\sqrt{a^2-x^2}-y}{\sqrt{a^2-y^2}-x} \cdot \frac{1}{\chi}|m_{xy}| \qquad (41)$$

From this expression, χ can be determined:

$$0 = \chi^2|m_{xy}| + \left(m_y - \frac{\sqrt{a^2-x^2}-y}{\sqrt{a^2-y^2}-x} m_x\right) \cdot \chi - \frac{\sqrt{a-x^2}-y}{\sqrt{a^2-y^2}-x} \cdot |m_{xy}| \qquad (42)$$

$$0 = \chi^2 + \frac{\left(m_y - \frac{\sqrt{a^2-x^2}-y}{\sqrt{a^2-y^2}-x} \cdot m_x\right)}{|m_{xy}|} \cdot \chi - \frac{\sqrt{a^2-x^2}-y}{\sqrt{a^2-y^2}-x} \qquad (43)$$

meaning:

$$\chi = -\frac{1}{2}\frac{\left(m_y - \frac{\sqrt{a^2-x^2}-y}{\sqrt{a^2-y^2}-x} \cdot m_x\right)}{|m_{xy}|} + \frac{1}{2}\sqrt{\left(\frac{\left(m_y - \frac{\sqrt{a^2-x^2}-y}{\sqrt{a^2-y^2}-x} m_x\right)}{m_{xy}^2}\right)^2 + 4 \cdot \frac{\sqrt{a^2-x^2}-y}{\sqrt{a^2-y^2}-x}} \qquad (44)$$

That is, with this value of χ, m_{sx} and m_{sy} corresponding to (15) and (16) are determined in the zone which is located within a distance from the edge of the point foundation less than the anchorage length. Although the point foundation is isotropically reinforced, it thus behaves at the edge as anisotropically reinforced with a linearly decreasing moment capacity towards the edge in each of the two directions.

With the formula for χ given by (21)/(22), the orientation of the inclined compression from the torsional moment is found in the areas where the reinforcement in both directions is fully anchored (at the center of the foundation). Formula (43) gives the orientation in the areas where the reinforcement in both directions are not fully anchored. In addition there are the two cases where only the reinforcement in one direction is fully anchored while the reinforcement in the other direction is not fully anchored.

Compared to formula (44), the term $\frac{\sqrt{a^2-x^2}-y}{\sqrt{a^2-y^2}-x}$ must be replaced by respectively:

$$\frac{\sqrt{a^2-x^2}-y}{l_b} \text{ for } \sqrt{a^2-x^2}-y > l_{b,max} \wedge \sqrt{a^2-y^2}-x \le l_{b,max} \qquad (45)$$

$$\frac{l_b}{\sqrt{a^2-y^2}-x} \text{ for } \sqrt{a^2-x^2}-y \le l_{b,max} \wedge \sqrt{a^2-y^2}-x > l_{b,max} \qquad (46)$$

With this approach, $T_{sx}/(P/2\pi z)$ shown in figure 5 changes to the variation shown in figure 8. The variation is shown for D/a = 0.1

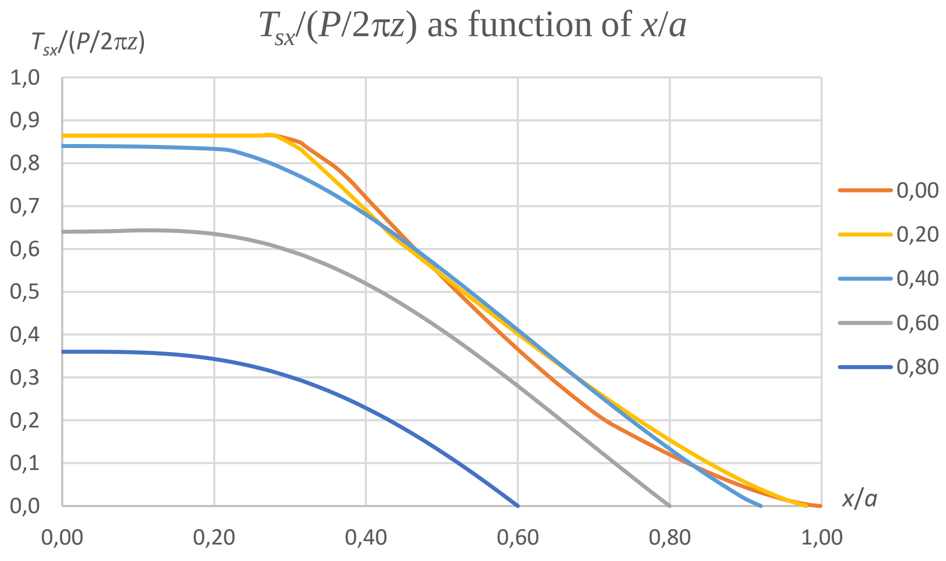


Figure 8 Tensile force taking into account the anisotropic behavior near the edge

Figure 9 shows a line with a slope of 1.43 for *y*/*a* = 0.4, which is the ratio of *y*/*a* that is most critical with respect to anchoring. It can be seen that this provides sufficient anchoring without anchored/bent-up reinforcement. This is illustrated in figure 9 for *D*/*a* = 0.1, and is as a very good approximation constant for different values of *D*/*a*.

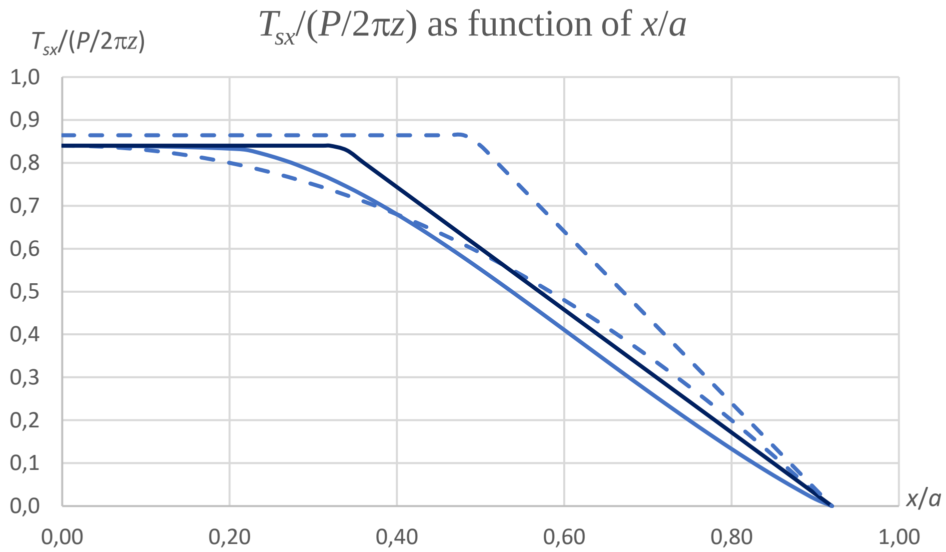


Figure 9 Illustration of the increased maximum anchorage length taking into account the anisotropic behavior near the edge

For comparison, the dashed line shows the extent of the required anchoring length without using the modified value of χ at the edge (see also figure 6).
Maximum anchoring length to avoid bent-up reinforcement is thus changed from (38) to:

$$l_{b,max} = \frac{T_{max}}{\left(\frac{dT}{dx}\right)_{max}} = \frac{1}{1{,}43} \cdot \left(1 - \sqrt[3]{\frac{D^2}{4a^2}}\right) a \approx 0{,}7 \cdot \left(1 - \sqrt[3]{\frac{D^2}{4a^2}}\right) a \qquad (47)$$

$$l_{b,max} = 1{,}4 \cdot \frac{1}{2}\left(1 - \sqrt[3]{\frac{D^2}{4a^2}}\right) a \qquad (48)$$

By taking into account the anisotropic conditions at the edge, an increase og 40% in the maximum anchorage length for the circular foundation is thus obtained.

Concluding remarks

This article establishes the maximum anchorage length for the reinforcement in the bottom side of a circular point foundation, with the aim of avoiding the use of bent-up reinforcement. Assuming that the foundation is equipped with orthotropic reinforcement with a constant degree of reinforcement in all sections, the necessary anchorage length for the reinforcement in a point foundation subjected by a centrally located normal force acting as a point load is first investigated. The necessary anchorage length for a point foundation subjected by a centrally located normal force acting over a finite area is then investigated. The modeling of the anchoring of the reinforcement as linearly growing from the end of the reinforcement means that the foundation will appear anisotropically reinforced near the edge, which is also treated in the article. The maximum anchorage length to achieve sufficient anchorage of the reinforcement is thus given by formula (48).

Literature

[1] Lars German Hagsten: ”Point Foundation. Lower Bound Solution for Distributed Load”. Proceedings of the Danish Society for Structural Science and Engineering. No. 1, May 2023, page 1-14.

[2] M.P. Nielsen & L. C. Hoang: 'Limit Analysis and Concrete Plasticity'. Third edition. 2011. CRC Press

Appendix A. Transformation formulas

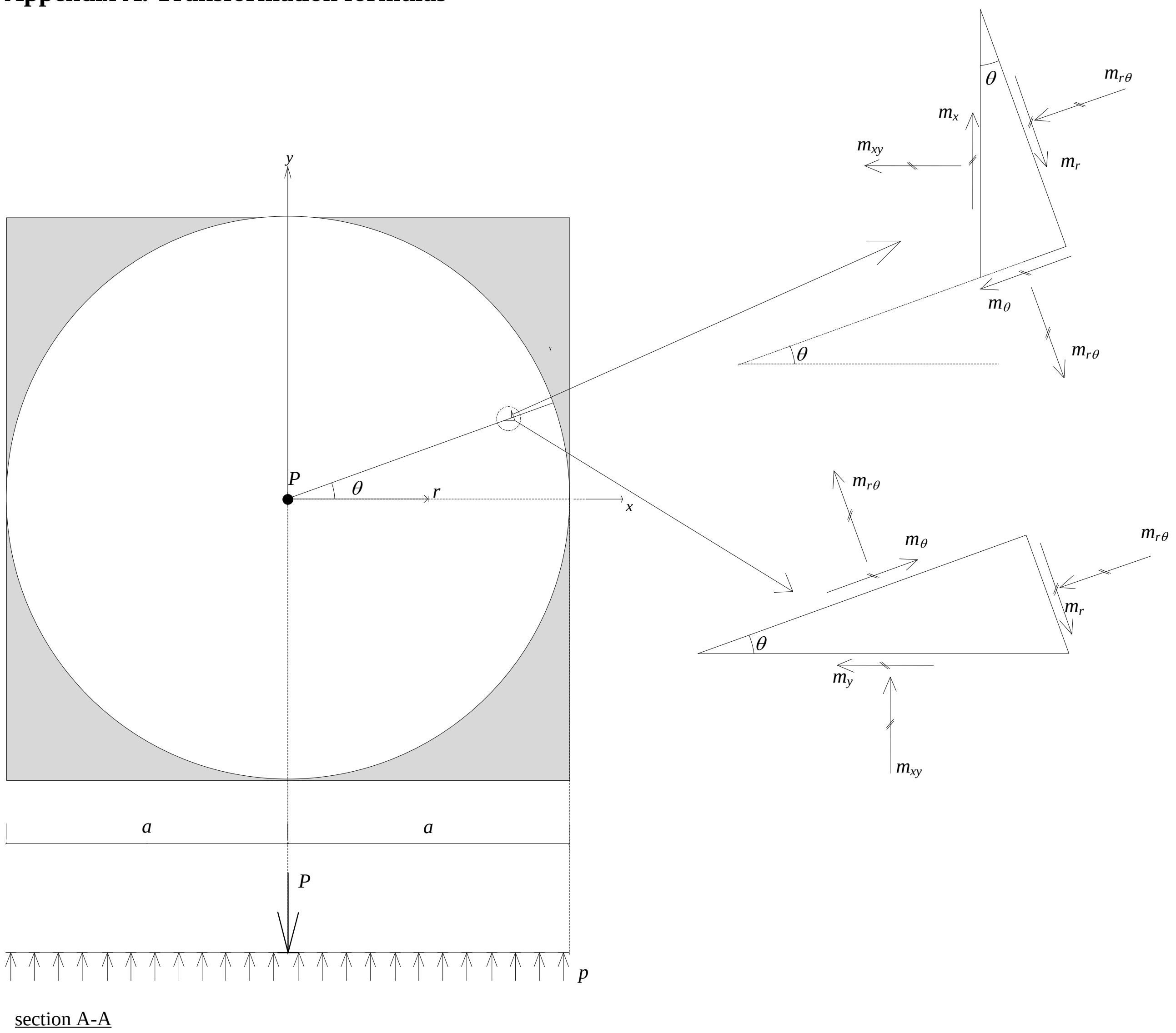


Figure A.1

$$m_x = m_r \cdot \cos^2\theta + m_\theta \cdot \sin^2\theta + 2m_{r\theta} \cdot \sin\theta \cdot \cos\theta \quad (A.1)$$

$$m_y = m_r \cdot \sin^2\theta + m_\theta \cdot \cos^2\theta - 2m_{r\theta} \cdot \sin\theta \cdot \cos\theta \quad (A.2)$$

$$m_{xy} = m_r \cdot \sin\theta \cdot \cos\theta - m_\theta \cdot \sin\theta \cdot \cos\theta + m_{r\theta}(\sin^2\theta - \cos^2\theta) \quad (A.3)$$

with

$$\theta = \arctan\left(\frac{y}{x}\right) \quad (A.4)$$

$$m_x = m_r \cdot \cos^2\left(\arctan\left(\frac{y}{x}\right)\right) + m_\theta \cdot \sin^2\left(\arctan\left(\frac{y}{x}\right)\right) + 2m_{r\theta} \cdot \sin\left(\arctan\left(\frac{y}{x}\right)\right) \cdot \cos\left(\arctan\left(\frac{y}{x}\right)\right) \quad (A.5)$$

$$m_y = m_r \cdot \sin^2\left(\arctan\left(\frac{y}{x}\right)\right) + m_\theta \cdot \cos^2\left(\arctan\left(\frac{y}{x}\right)\right) - 2m_{r\theta} \cdot \sin\left(\arctan\left(\frac{y}{x}\right)\right) \cdot \cos\left(\arctan\left(\frac{y}{x}\right)\right) \quad (A.6)$$

$$m_{xy} = (m_r - m_\theta) \cdot \sin\left(\arctan\left(\frac{y}{x}\right)\right) \cdot \cos\left(\arctan\left(\frac{y}{x}\right)\right) + m_{r\theta}\left(\sin^2\left(\arctan\left(\frac{y}{x}\right)\right) - \cos^2\left(\arctan\left(\frac{y}{x}\right)\right)\right) \quad (A.7)$$

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